

Advances in Latent Variable Modeling Using Mplus Version 7

Bengt Muthén

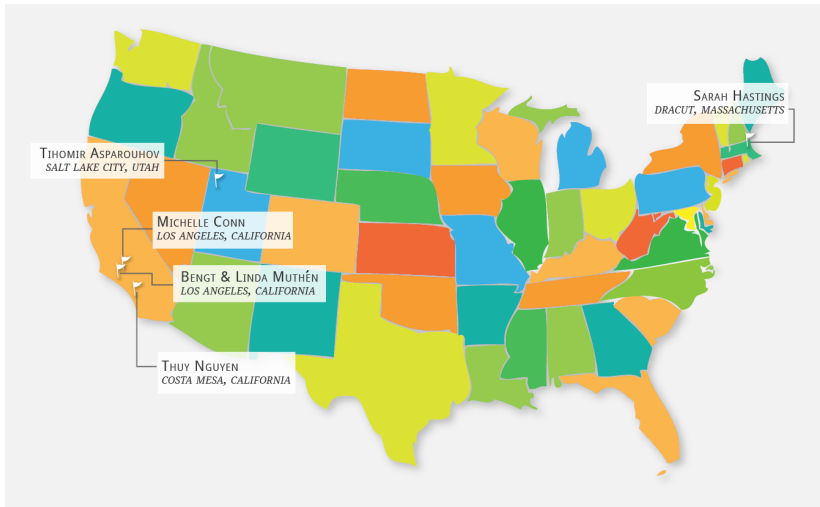
Mplus

www.statmodel.com

bmuthen@statmodel.com

Workshop at the Modern Modeling Methods Conference,
University of Connecticut, May 23, 2013
and at the APS Convention, Washington DC, May 24, 2013

The Map of the Mplus Team



The Creative Force Behind Mplus: Thuy, Linda, Bengt, Tihomir (in the Offices, c:a 2001)



Several programs in one

- Path analysis
- Exploratory factor analysis
- Structural equation modeling
- Item response theory analysis
- Growth modeling
- Mixture modeling (latent class analysis)
- Longitudinal mixture modeling (Markov, LTA, LCGA, GMM)
- Survival analysis (continuous- and discrete-time)
- Multilevel analysis
- Complex survey data analysis
- Bayesian analysis
- Monte Carlo simulation

Fully integrated in a general latent variable framework

What's New in Mplus Version 7?

5 big new features:

- 1 Diagrammer
- 2 Factor analysis
 - Bi-factor EFA rotations, bi-factor ESEM, two-tier modeling
 - Bayesian EFA and CFA (BSEM), bi-factor BSEM
- 3 Analysis of several groups with approx. measurement invariance
 - using a Bayes approach (multiple-group BSEM)
 - using a two-level analysis with random intercepts and loadings
- 4 Analysis of individual differences SEM using measurement parameters that vary across subjects
- 5 Mixture analysis
 - Using a proper 3-step analyze-classify-analyze approach to investigate covariates and distal outcomes
 - Latent transition analysis with new output, covariates influencing transition probabilities, and probability parameterization
 - Exploratory LCA using Bayesian analysis

5 more big features:

- 1 3-level SEM analysis, complex survey data handling, and multiple imputation
- 2 Cross-classified SEM analysis including random subjects and contexts (2 random modes)
- 3 IRT analysis with random items
- 4 Longitudinal analysis with approx. measurement invariance
 - using a Bayes approach (multiple-time point BSEM)
 - using cross-classified analysis of time and subjects
- 5 Analysis of changing membership over time

and 5 other new features:

- 1 Parallel analysis
- 2 LOOP plots (moderated mediation, cross-level interactions, etc)
- 3 Bayes plausible value factor score distribution plots for each subject
- 4 Two-tier algorithm
- 5 New convenience options: LOOP, DO, COV, DIFF, DO DIFF, MODEL=ALLFREE, auto-labeling, BY with random loadings, BITER = (minimum), TECH15, TECH16

What's New In Mplus Version 7.1?

Version 7.1 contains corrections to minor problems that have been found since the release of Version 7 in September 2012 as well as the following new features:

- 1 Multiple-group factor analysis: A new method
- 2 Multiple-group factor analysis: Convenience features
- 3 Exploratory factor analysis: Convenience features
- 4 Mixture modeling: 3-step modifications
- 5 Mixture modeling: A new distal outcome stepwise method
- 6 New TECH4 output
- 7 GROUPING and KNOWNCLASS convenience features
- 8 DO option for MODEL TEST

For more information, see Version History at the Mplus web site www.statmodel.com, including 22 new User's Guide examples

Videos and pdfs from the Mplus Version 7 training at Utrecht University August 27-29 can be found via the Mplus home page

Example: Two groups, 9 factor loadings in each group, expressing the 9 group differences (note that the longer symbol – denotes a "dash", namely a list, and the shorter symbol - denotes minus)

DO(1,9) diff# = lambda1# – lambda2#;

! Same as

! diff1 = lambda11 - lambda21;

! diff2 = lambda12 - lambda22:

! ...

! diff9 = lambda19 - lambda29;

- Useful in MODEL CONSTRAINT to create NEW parameters
- Useful in MODEL PRIORS for Bayesian analysis
- Can also be used in DEFINE and MODEL TEST (Version 7.1)

ANALYSIS:

TYPE = TWOLEVEL RANDOM;

ESTIMATOR = BAYES;

MODEL:

% WITHIN %

s1-s10 | f BY y1-y10;

% BETWEEN %

[s1-s10];

s1-s10;

Easier than a series of statements like:

s | y ON f;

1. Mediation Analysis

- LOOP plot for moderated mediation
- Causal effects

1.1 LOOP Option

LOOP is used in MODEL CONSTRAINT in conjunction with the PLOT option to create plots of one variable related to another, including a 95% confidence interval. An example:

MODEL: y ON x (p1);

MODEL CONSTRAINT:

PLOT(ypred);

LOOP(age, 20, 50, 1); ! 20 ≤ age ≤ 50 with steps of 1

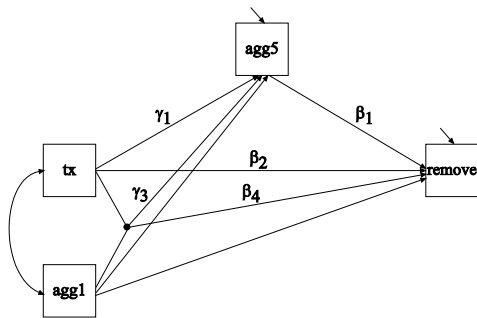
ypred = p1*age;

- Plotting indirect effects with moderated mediation. Preacher, Rucker, Hayes (2007), MBR: Figure 3 - conditional indirect effect as a function of the moderator
- Plotting cross-level interactions in two-level modeling. Bauer & Curran (2005)
- Plotting sensitivity graphs for causal effect mediation modeling. Imai et al. (2010), Psych Methods; Muthén (2011)

1.2 Example: Moderated Mediation Of Aggressive Behavior

- Randomized field experiment in Baltimore public schools
- Classroom-based intervention in Grade 1 aimed at reducing aggressive-disruptive classroom behavior among elementary school students
- The variable `agg1` represents the pre-intervention aggression score in Grade 1 used as a covariate in the analysis to strengthen the power to detect treatment effects
- `Agg 1` also serves to explore a hypothesis of treatment-baseline interaction using the interaction between the treatment dummy variable `tx` and `agg1`, labeled `inter`. The `agg1` covariate is referred to as a moderator
- The mediator variable `agg5` is the Grade 5 aggression score
- The distal outcome variable `remove` is the number of times the student has been removed from school
- The analysis is based on $n = 392$ boys in treatment and control classrooms

Example: Moderated Mediation of Aggressive Behavior



$$remove = \beta_0 + \beta_1 agg5 + \beta_2 tx + \beta_3 agg1 + \beta_4 tx agg1 + \varepsilon_1, \quad (1)$$

$$agg5 = \gamma_0 + \gamma_1 tx + \gamma_2 agg1 + \gamma_3 tx agg1 + \varepsilon_2, \quad (2)$$

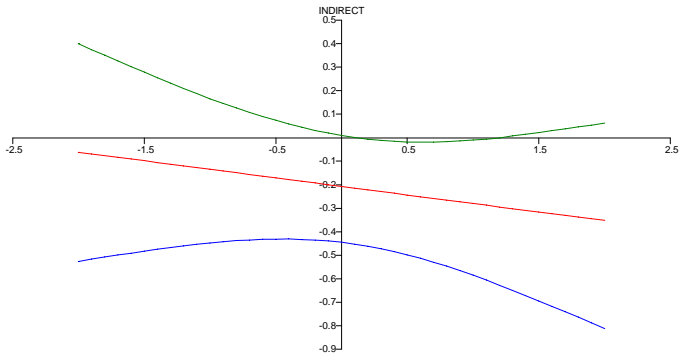
$$= \gamma_0 + (\gamma_1 + \gamma_3 agg1) tx + \gamma_2 agg1 + \varepsilon_2. \quad (3)$$

Indirect effect of tx on remove is $\beta_1 (\gamma_1 + \gamma_3 agg1)$, where agg1 moderates the effect of the treatment. Direct effect: $\beta_2 + \beta_4 agg1$.

Input For Moderated Mediation Of Aggressive Behavior

```
DEFINE:      inter = tx*agg1;
ANALYSIS:   ESTIMATOR = BAYES;
            PROCESSORS = 2; BITERATIONS = (30000);
MODEL:      remove ON agg5 (beta1)
            tx (beta2)
            agg1 (beta3)
            inter (beta4);
            agg5 ON tx (gamma1)
            agg1 (gamma2)
            inter (gamma3);
MODEL CONSTRAINT:
            PLOT(indirect direct);
            ! let moderate represent the range of the agg1 moderator
            LOOP(moderate, -2, 2, 0.1);
            indirect = beta1*(gamma1+gamma3*moderate);
            direct = beta2+beta4*moderate;
PLOT:       TYPE = PLOT2;
```


LOOP Plot: Indirect Effect Of Treatment Plotted as a Function of a Moderator



See Mplus User's Guide ex 3.18

Muthén (2011). Applications of Causally Defined Direct and Indirect Effects in Mediation Analysis using SEM in Mplus.

The paper, an appendix with formulas, and Mplus scripts are available at www.statmodel.com under Papers, Mediational Modeling

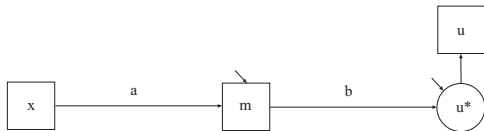
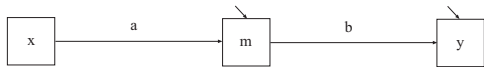
New ways to estimate mediation effects with categorical and other non-normal mediators and distal outcomes

Causally-Defined Indirect and Direct Effects

- Causally-defined effects based on counterfactuals and potential outcomes using expectations have been developed by Robins, Greenland, Pearl, VanderWeele, Imai etc
 - Total indirect effect (TIE) and pure indirect effect (PIE)
 - Direct effect (DE)
 - Total effect (TE)
- Same results for SEM with continuous outcomes (but a clearer way of defining them)
- Different results than SEM with for instance categorical DVs
 - Causal effect researchers say that SEM results such as the indirect effect $a \times b$ are wrong
- Most SEM users are not aware of this development

The difference between the causal effects and the effects obtained by what is called the naive approach has been studied in Imai et al. (2010a) and Pearl (2011c). Imai et al. (2010a, Appendix E, p. 23) conducted a Monte Carlo simulation study to show the biases, while Pearl (2011c) presented graphs showing the differences.

Continuous versus Binary Distal Outcome

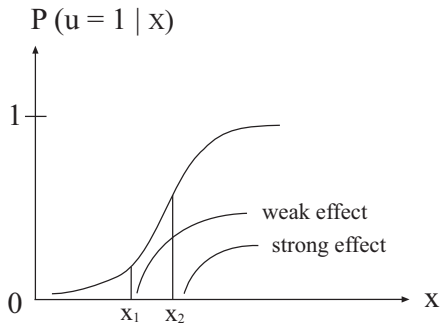


Conventional versus Causal Mediation Effects with a Categorical Distal Outcome

With a categorical distal outcome, the conventional product formula for an indirect effect is valid only for an underlying continuous latent response variable behind the categorical observed outcome (2 linear regressions), not for the observed categorical outcome itself (linear plus non-linear regression).

Similarly, with a categorical mediator, conventional product formulas for indirect effects are only relevant/valid for a continuous latent response variable behind the mediator.

The Problem with $a \times b$ for a Binary Outcome: 2 Parameters used when 5 are Needed



- The $a \times b$ indirect effect faces the problem of non-constant effect due to ignoring the level parameters (the intercept for the mediator and threshold for the distal outcome)
- The causally-defined indirect effect uses these level parameters, focusing on the expected values of the observed binary outcome - the probabilities

The Probability of $u = 1|x$

- Conditional on x , $P(u = 1|x)$ is obtained by integrating over the residual of the mediator and apart from the regression coefficients the probability involves the residual variance, the mediator intercept, and the distal outcome threshold
- One can compute $P(u = 1|x = 1) - P(u = 1|x = 0)$ to compare treatment and control
- This is TE (Total Effect)
- But what are the indirect and direct effects?
- This is where the counterfactual definitions of casual effects come in

The causal effects TIE, PIE, DE, and TE are expressed in a general way and can be applied to many different settings:

- Continuous mediator, continuous distal outcome (gives the usual SEM formulas)
- Categorical mediator, continuous distal outcome
- Continuous mediator, categorical distal outcome
- Categorical mediator, categorical distal outcome
- Count distal outcome
- Nominal mediator
- Survival distal outcome

- The direct and indirect effects can be estimated in Mplus using maximum-likelihood or Bayes
- ML:
 - Standard errors of the direct and indirect causal effects are obtained by the delta method using the Mplus MODEL CONSTRAINT command
 - Bootstrapped standard errors and confidence intervals are also available, taking into account possible non-normality of the effect distributions
- Bayes:
 - Bayesian analysis is available in order to describe the possible non-normal posterior distributions

Muthén (2011). Applications of Causally Defined Direct and Indirect Effects in Mediation Analysis using SEM in Mplus

2. Bayesian Analysis

- Bayesian analysis firmly established and its use is growing in mainstream statistics
- Much less use of Bayes outside statistics
- Bayesian analysis not sufficiently accessible in other programs
- Bayesian analysis was introduced in Mplus Version 6 and greatly expanded in Version 7: Easy to use
- Bayes provides a broad platform for further Mplus development

Why do we have to learn about Bayes?

- More can be learned about parameter estimates and model fit
- Better small-sample performance, large-sample theory not needed
- Non-informative versus informative priors
- Frequentists can see Bayes with non-informative priors as a computing algorithm to get answers that would be the same as ML if ML could have been done
- Informative priors can better reflect substantive hypotheses
- Analyses can be made less computationally demanding
- New types of models can be analyzed

For a Bayes introduction with further references, see, e.g., Muthén (2010). Bayesian analysis in Mplus: A brief introduction. Technical Report. Version 3.

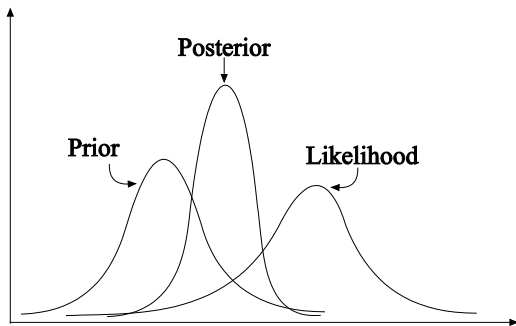
Writings On The Bayes Implementation In Mplus

- Muthén (2010). Bayesian analysis in Mplus: A brief introduction. Technical Report. Version 3.
- Asparouhov & Muthén (2010). Bayesian analysis using Mplus: Technical implementation. Technical Report. Version 3.
- Asparouhov & Muthén (2010). Bayesian analysis of latent variable models using Mplus. Technical Report. Version 4.
- Asparouhov & Muthén (2010). Multiple imputation with Mplus. Technical Report. Version 2.
- Asparouhov & Muthén (2010). Plausible values for latent variable using Mplus. Technical Report.
- Muthén & Asparouhov (2012). Bayesian SEM: A more flexible representation of substantive theory. Psychological Methods
- Asparouhov & Muthén (2011). Using Bayesian priors for more flexible latent class analysis.
- Asparouhov & Muthén (2012). General random effect latent variable modeling: Random subjects, items, contexts, and parameters.
- Asparouhov & Muthén (2012). Comparison of computational methods for high dimensional item factor analysis.

Posted under Papers, Bayesian Analysis and Latent Class Analysis

Prior, Likelihood, And Posterior

- Frequentist view: Parameters are fixed. ML estimates have an asymptotically-normal distribution
- Bayesian view: Parameters are variables that have a prior distribution. Estimates have a possibly non-normal posterior distribution. Does not depend on large-sample theory
 - Non-informative (diffuse) priors vs informative priors



Bayesian Estimation Obtained Iteratively

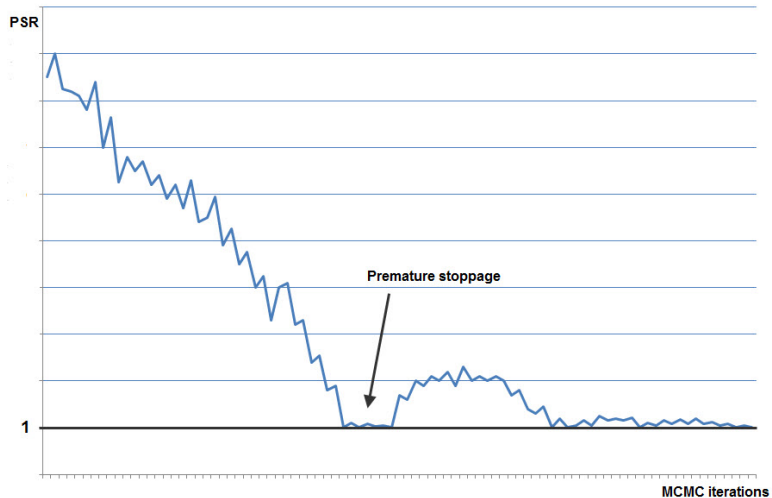
Using Markov Chain Monte Carlo (MCMC) Algorithms

- θ_i : vector of parameters, latent variables, and missing observations at iteration i
- θ_i is divided into S sets:
$$\theta_i = (\theta_{1i}, \dots, \theta_{Si})$$
- Updated θ using Gibbs sampling over $i = 1, 2, \dots, n$ iterations:
$$\theta_{1i} | \theta_{2i-1}, \dots, \theta_{Si-1}, \text{ data, priors}$$
$$\theta_{2i} | \theta_{3i-1}, \dots, \theta_{Si-1}, \text{ data, priors}$$
$$\dots$$
$$\theta_{Si} | \theta_{1i}, \dots, \theta_{S-1i-1}, \text{ data, priors}$$

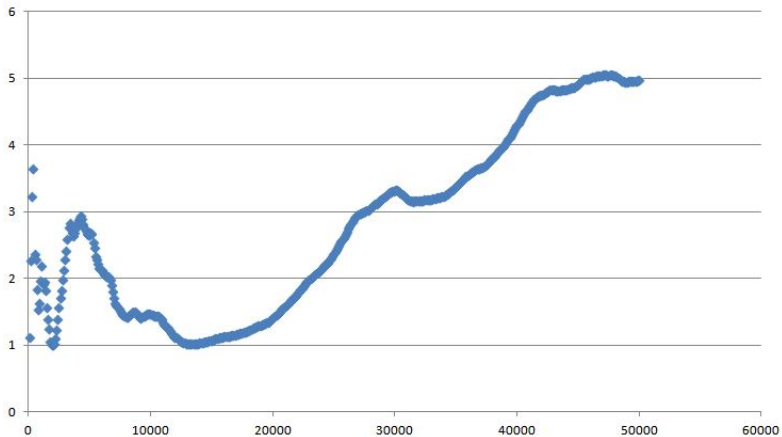
Asparouhov & Muthén (2010). Bayesian analysis using Mplus. Technical implementation. Technical Report.

- Trace plot: Graph of the value of a parameter at different iterations
- Burnin phase: Discarding early iterations. Mplus discards first half
- Posterior distribution: Mplus uses the last half as a sample representing the posterior distribution
- Autocorrelation plot: Correlation between consecutive iterations for a parameter. Low correlation desired
- Mixing: The MCMC chain should visit the full range of parameter values, i.e. sample from all areas of the posterior density
- Convergence: Stationary process
- Potential Scale Reduction (PSR): Between-chain variation small relative to total variation. Convergence when $PSR \approx 1$

PSR Convergence Issues: Premature Stoppage

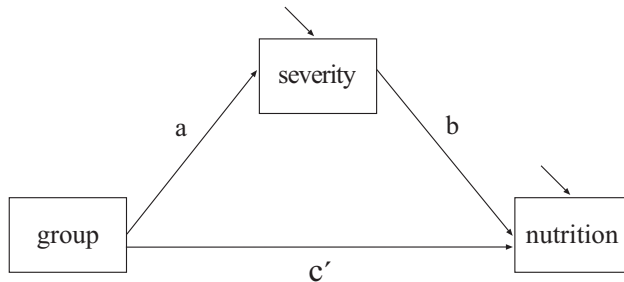


PSR Convergence Issues: Premature Stoppages Due to Non-Identification



Bayesian Mediation Modeling With Non-Informative Priors: The MacKinnon ATLAS Example

Source: MacKinnon et al. (2004), *Multivariate Behavioral Research*.
 $n = 861$



- Intervention aimed at increasing perceived severity of using steroids among athletes. Perceived severity of using steroids is in turn hypothesized to increase good nutrition behaviors
- Indirect effect: $a \times b$

Input For Bayesian Analysis Of ATLAS Example Using The Default Of Non-Informative Priors

TITLE: ATLAS
DATA: FILE = mbr2004atlast.txt;
VARIABLE: NAMES = obs group severity nutrit;
USEVARIABLES = group - nutrit;
ANALYSIS: ESTIMATOR = BAYES;
PROCESSORS = 2;
BITERATIONS = (10000); ! minimum of 10K iterations
MODEL: severity ON group (a);
nutrit ON severity (b)
group;
MODEL CONSTRAINT:
NEW (indirect);
indirect = a*b;
OUTPUT: TECH1 TECH8 STANDARDIZED;
PLOT: TYPE = PLOT2;

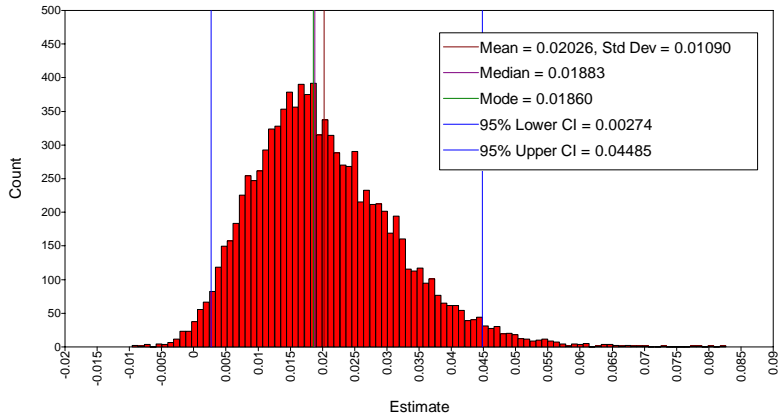
Output For Bayesian Analysis Of ATLAS Example

Parameter	Estimate	Posterior	One-Tailed	95% C.I.	
		S.D.	P-Value	Lower 2.5%	Upper 2.5%
severity ON					
group	0.272	0.089	0.001	0.098	0.448
nutrit ON					
severity	0.074	0.030	0.008	0.014	0.133
group	-0.018	0.080	0.408	-0.177	0.140
Intercepts					
severity	5.648	0.062	0.000	5.525	5.768
nutrit	3.663	0.177	0.000	3.313	4.014

Output For Bayesian Analysis Of ATLAS Example (Continued)

Parameter	Estimate	Posterior	One-Tailed	95% C.I.	
		S.D.	P-Value	Lower 2.5%	Upper 2.5%
Residual variances					
severity	1.719	0.083	0.000	1.566	1.895
group	1.333	0.065	0.000	1.215	1.467
New/additional parameters					
indirect	0.019	0.011	0.009	0.003	0.045

Bayesian Posterior Distribution For The Indirect Effect



Bayesian Posterior Distribution For The Indirect Effect: Conclusions

- Bayesian analysis: There is a mediated effect of the intervention
 - The 95% Bayesian credibility interval does not include zero
- ML analysis: There is not a mediated effect of the intervention
 - ML-estimated indirect effect is not significantly different from zero and the symmetric confidence interval includes zero
 - Bootstrap SEs and CIs can be used with ML

3. Factor Analysis

Types of factor analyses in Mplus:

- Exploratory Factor Analysis (EFA): Regular and bi-factor rotations
- Confirmatory Factor Analysis (CFA)
- Exploratory Structural Equation Modeling (ESEM; Asparouhov & Muthén, 2009 in Structural Equation Modeling)
- Bayesian Structural Equation Modeling (BSEM; Muthén & Asparouhov, 2012 in Psychological Methods)

- Single-group analysis
 - Parallel analysis to decide on the number of factors
 - Bi-factor EFA rotations
 - Bayesian factor analysis
 - Bayesian EFA
 - Bayesian factor scores (plausible values)
 - Bayesian CFA (BSEM)
 - Two-tier modeling for confirmatory bi-factor analysis
- Analysis of several groups such as cross-cultural comparisons
 - Multiple-group BSEM
 - Two-level analysis with random intercepts and loadings
 - Invariance alignment optimization

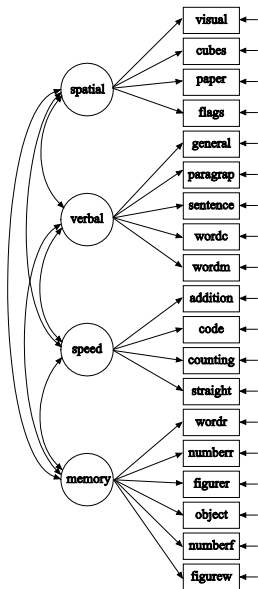
3.2 EFA Of Holzinger-Swineford Mental Abilities Data

- Classic 1939 factor analysis study by Holzinger and Swineford (1939) in Illinois schools
- Twenty-six tests intended to measure a general factor and five specific factors
- Administered to seventh and eighth grade students in two schools
 - Grant-White school ($n = 145$). Students came from homes where the parents were mostly American-born
 - Pasteur school ($n = 156$). Students came largely from working-class parents of whom many were foreign-born and where their native language was used at home
- Source:
 - Holzinger, K. J. & Swineford, F. (1939). A study in factor analysis: The stability of a bi- factor solution. Supplementary Educational Monographs. Chicago, Ill.: The University of Chicago

Current analyses:

- 19 variables using tests hypothesized to measure four mental abilities: Spatial, verbal, speed, and memory
- 24 variables, adding 5 tests measuring a general ability (deduction, test taking ability)

19 Variables: Expected Factor Loading Pattern

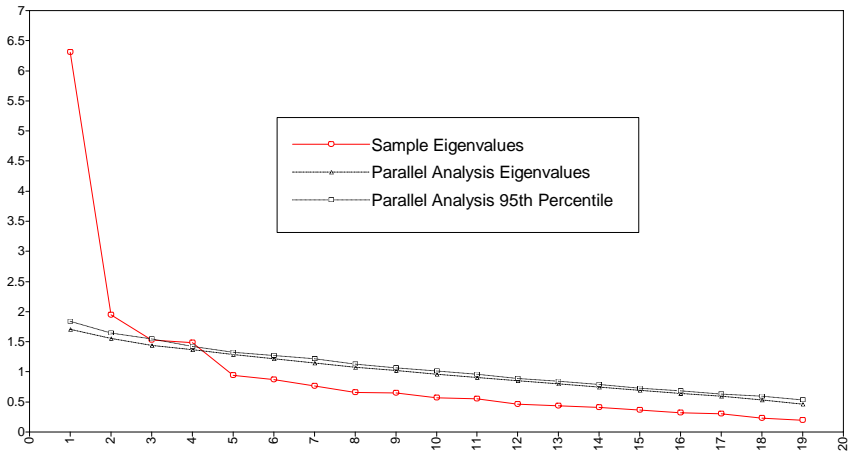


	Spatial	Verbal	Speed	Memory
visual	x	0	0	0
cubes	x	0	0	0
paper	x	0	0	0
flags	x	0	0	0
general	0	x	0	0
paragraf	0	x	0	0
sentence	0	x	0	0
wordc	0	x	0	0
wordm	0	x	0	0
addition	0	0	x	0
code	0	0	x	0
counting	0	0	x	0
straight	0	0	x	0
wordr	0	0	0	x
numberr	0	0	0	x
figurer	0	0	0	x
object	0	0	0	x
numberf	0	0	0	x
figurew	0	0	0	x

Holzinger-Swineford, 19 Variables: Input Excerpts For EFA

```
VARIABLE:  USEVARIABLES = visual - figurew;  
           USEOBSERVATIONS = school EQ 0;  
ANALYSIS:  TYPE = EFA 1 6;  
           ROTATION = GEOMIN; ! default  
           ESTIMATOR = ML; ! default  
           PARALLEL = 50;  
OUTPUT:    SAMPSTAT MODINDICES;  
PLOT:      TYPE = PLOT3;
```

Parallel Analysis Of The Eigenvalues For 19-Variable Holzinger-Swineford, Grant-White EFA



EFA ML χ^2 Tests Of Model Fit For 19-Variable Holzinger-Swineford Data, Grant-White School

Factors	Chi-Square χ^2	df	p	BIC	CFI	RMSEA	SRMR
1	469.81	152	.000	18637	.68	.120	.102
2	276.44	134	.000	18534	.86	.086	.068
3	188.75	117	.000	18531	.93	.065	.053
4	110.34	101	.248	18532	.99	.025	.030
5	82.69	86	.581	18579	1.00	.000	.025
6	no convergence						

EFA ML Model Test Results

For 4-Factor, 19-Variable Holzinger-Swineford Data
For The Grant-White (n =145) And Pasteur (n=156) Schools

Model	χ^2	df	P-value	RMSEA	CFI
Grant-White					
EFA	110	101	0.248	0.025	0.991
Pasteur					
EFA	128	101	0.036	0.041	0.972

Estimated EFA factor pattern using oblique rotation with Geomin:
Grant-White has 6 and Pasteur has 9 significant cross-loadings.

	Grant-White Factor Loading Patterns For EFA				Pasteur Factor Loading Pattern For EFA			
	Spatial	Verbal	Speed	Memory	Spatial	Verbal	Speed	Memory
visual	0.628*	0.065	0.091	0.085	0.580*	0.307*	-0.001	0.053
cubes	0.485*	0.050	0.007	-0.003	0.521*	0.027	-0.078	-0.059
paper	0.406*	0.107	0.084	0.083	0.484*	0.101	-0.016	-0.229*
flags	0.579*	0.160	0.013	0.026	0.687*	-0.051	0.067	0.101
general	0.042	0.752*	0.126	-0.051	-0.043	0.838*	0.042	-0.118
paragrap	0.021	0.804*	-0.056	0.098	0.026	0.800*	-0.006	0.069
sentence	-0.039	0.844*	0.085	-0.057	-0.045	0.911*	-0.054	-0.029
wordc	0.094	0.556*	0.197*	0.019	0.098	0.695*	0.008	0.083
wordm	0.004	0.852*	-0.074	0.069	0.143*	0.793*	0.029	-0.023
addition	-0.302*	0.029	0.824*	0.078	-0.247*	0.067	0.664*	0.026
code	0.012	0.050	0.479*	0.279*	0.004	0.262*	0.552*	0.082
counting	0.045	-0.159	0.826*	-0.014	0.073	-0.034	0.656*	-0.166
straight	0.346*	0.043	0.570*	-0.055	0.266*	-0.034	0.526*	-0.056
wordr	-0.024	0.117	-0.020	0.523*	-0.005	0.020	-0.039	0.726*
numberr	0.069	0.021	-0.026	0.515*	-0.026	-0.057	-0.057	0.604*
figurer	0.354*	-0.033	-0.077	0.515*	0.329*	0.042	0.168	0.403*
object	-0.195	0.045	0.154	0.685*	-0.123	-0.005	0.333*	0.469*
numberf	0.225	-0.127	0.246*	0.450*	-0.014	0.092	0.092	0.427*
figurew	0.069	0.099	0.058	0.365*	0.139	0.013	0.237*	0.291*

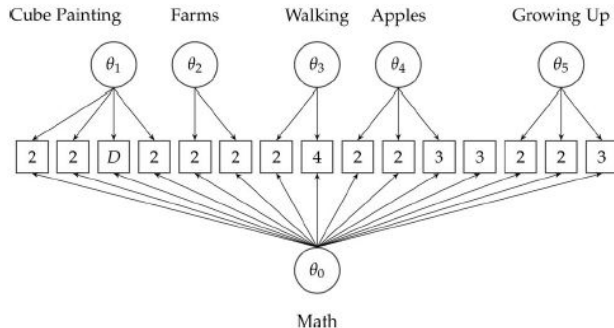
3.3 Bi-Factor Modeling Overview

- General factor influencing all items (deductive, test-taking ability); Holzinger-Swineford (1939) 24-variable model
- Testlet modeling, e.g. for PISA test items
- Longitudinal modeling with across-time correlation for residuals

Bi-factor modeling is as popular today as in 1939. New developments for faster maximum-likelihood estimation with categorical items, reducing the number of dimensions for numerical integration:

- Gibbons, & Hedeker (1992). Full-information item bi-factor analysis. *Psychometrika*
- Reise, Morizot, & Hays (2007). The role of the bifactor model in resolving dimensionality issues in health outcomes measures. *Quality of Life Research*
- Cai (2010). A two-tier full-information item factor analysis model with applications. *Psychometrika*
- Cai, Yang, Hansen (2011). Generalized full-information item bifactor analysis. *Psychological Methods*

Bi-Factor Model For PISA Math Items

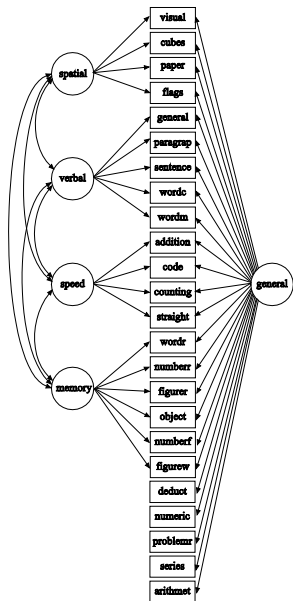


With categorical items, a two-tier algorithm for ML reduces the 6 dimensions of integration to 2.

Cai, Yang, & Hansen (2011) Generalized full-information item bifactor analysis. *Psychological Methods*, 16, 221-248

- Bi-factor EFA (Jennrich & Bentler, 2011, 2012, Psychometrika)
 - Allowing a general factor that influences all variables
 - ROTATION = BI-GEOMIN (new in Mplus Version 7)
 - Muthén & Asparouhov (2013). Item response modeling in Mplus: A multi-dimensional, multi-level, and multi-timepoint example. Forthcoming in Linden & Hambleton (2013). Handbook of item response theory: Models, statistical tools, and applications
- Bi-factor ESEM (Exploratory Structural Equation Modeling)
 - ROTATION = BI-GEOMIN (same as above)
 - Bi-factor ESEM with a general CFA factor and ROTATION = GEOMIN for specific factors
- Bi-factor BSEM (Bayesian SEM)
 - No rotation
 - Less rigid version of CFA bi-factor analysis

Holzinger-Swineford 24-variable bi-factor example:



	General	Spatial	Verbal	Speed	Memory
visual	x	x	0	0	0
cubes	x	x	0	0	0
paper	x	x	0	0	0
flags	x	x	0	0	0
general	x	0	x	0	0
paragraf	x	0	x	0	0
sentence	x	0	x	0	0
wordc	x	0	x	0	0
wordm	x	0	x	0	0
addition	x	0	0	x	0
code	x	0	0	x	0
counting	x	0	0	x	0
straight	x	0	0	x	0
words	x	0	0	0	x
numberr	x	0	0	0	x
figurer	x	0	0	0	x
object	x	0	0	0	x
numberf	x	0	0	0	x
figurew	x	0	0	0	x
deduct	x	0	0	0	0
numeric	x	0	0	0	0
problemr	x	0	0	0	0
series	x	0	0	0	0
arithmet	x	0	0	0	0

Bi-Factor Modeling Of The 24-Variable Holzinger-Swineford Data: Input Excerpts For Bi-Factor EFA

Requesting one general factor and four specific factors:

```
VARIABLE:  USEVARIABLES = visual - arithmet;  
           USEOBSERVATIONS = school EQ 0;  
ANALYSIS:  TYPE = EFA 5 5;  
           ROTATION = BI-GEOMIN;
```

Bi-Factor EFA Solution For Holzinger-Swineford's 24-Variable Grant-White Data

	General	Spatial	Verbal	Speed	Memory
visual	0.621*	0.384*	-0.065	0.072	0.002
cubes	0.433*	0.207	-0.103	-0.115	-0.118
paper	0.430*	0.343*	0.058	0.225	0.079
flags	0.583*	0.311*	-0.028	-0.077	-0.109
general	0.610*	-0.034	0.524*	0.001	-0.075
paragrap	0.554*	0.053	0.618*	0.012	0.102
sentence	0.572*	-0.037	0.622*	0.010	-0.064
wordc	0.619*	0.006	0.354*	0.038	-0.048
wordm	0.582*	-0.008	0.603*	-0.137	0.009
addition	0.508*	-0.528	-0.036	0.327	0.009
code	0.532*	-0.031	0.046	0.428*	0.310*
counting	0.568*	-0.229	-0.216*	0.302	-0.093
straight	0.643*	0.217	0.004	0.526*	-0.032

Bi-Factor EFA Results For Holzinger-Swineford, Continued

	General	Spatial	Verbal	Speed	Memory
wordr	0.349*	0.018	0.077	0.032	0.475*
numberr	0.352*	0.037	-0.041	-0.052	0.392*
figurer	0.495*	0.221	-0.122	-0.033	0.384*
object	0.422*	-0.200	-0.010	-0.021	0.497*
numberf	0.553*	-0.041	-0.220*	0.003	0.256*
figurew	0.414*	-0.033	-0.003	-0.024	0.246*
deduct	0.611*	-0.001	0.089	-0.284*	0.036
numeric	0.656*	-0.021	-0.129	0.029	-0.023
problemr	0.607*	0.028	0.091	-0.227*	0.059
series	0.714*	0.023	0.034	-0.202	-0.067
arithmet	0.638*	-0.356*	0.092	-0.009	0.070

6 significant cross-loadings

BI-GEOMIN Factor Correlations

	General	Spatial	Verbal	Speed	Memory
General	1.000				
Spatial	0.000	1.000			
Verbal	0.000	0.022	1.000		
Speed	0.000	-0.223*	-0.122*	1.000	
Memory	0.000	-0.037	0.068	-0.134	1.000

ML χ^2 test of model fit has p-value = 0.3043.

- Bi-factor EFA with 1 general and $m-1$ specific factors has the same model fit as regular EFA with m factors (same ML loglikelihood and number of parameters); it is just another rotation of the factors
- For the 24-variable Holzinger-Swineford data, bi-factor EFA with 1 general and 4 specific factors gives a simple factor pattern that largely agrees with the Holzinger-Swineford hypotheses
- In contrast, regular 5-factor EFA for the 24-variable Holzinger-Swineford data does not give a simple factor loading pattern

- Bayesian EFA
- Bayesian factor scores (plausible values)
- Bayesian CFA (BSEM)

3.6 Bayesian EFA

- Bayesian estimation of exploratory factor analysis implemented in Mplus version 7 for models with continuous and categorical variables
- Asymptotically the Bayes EFA is the same as the ML solution
- Bayes EFA for categorical variable is a full information estimation method without using numerical integration and therefore feasible with any number of factors
- Asparouhov and Muthén (2012). Comparison of computational methods for high dimensional item factor analysis
- For more on Bayesian EFA, see the Day1 handout and video from Utrecht, August 2012

3.7 Bayes Factor Scores Handling

- New improved language for factor scores with Bayesian estimation. The same language as for other estimators
- `SAVEDATA: FILE=fs.dat; SAVE=FS(300); FACTORS=factor names;` This command specifies that 300 imputations will be used to estimate the factor scores and that plausible value distributions are available for plotting
- Posterior mean, median, confidence intervals, standard error, all imputed values, distribution plot for each factor score for each latent variable for any model estimated with the Bayes estimator
- Bayes factor score advantages: more accurate than ML factor scores in small sample size, Bayes factor score more accurate in secondary analysis such as for example computing correlations between factor

3.8 Bayesian CFA (BSEM)

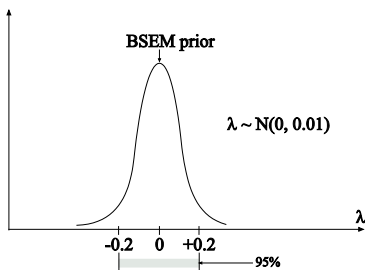
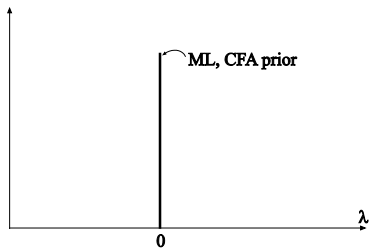
- Regular CFA is too strict, seldom fits well, and overestimates factor correlations
- Bayes CFA (BSEM) is more flexible, using zero-mean-small-variance informative priors to allow for cross-loadings, residual correlations, and direct effects which are not identified in ML

Muthén & Asparouhov (2012). Bayesian SEM: A more flexible representation of substantive theory. *Psychological Methods*, 17, 313-335. With commentaries and a rejoinder.

Golay, Reverte, Rossier, Favez & Lecerf (2012, November 12). Further insights on the French WISCIV factor structure through Bayesian structural equation modeling. *Psychological Assessment*. Advance online publication. DOI: 10.1037/a0030676

ML versus BSEM Priors

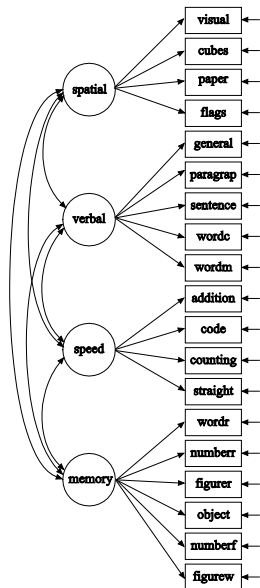
- ML CFA is characterized by many zero factor loadings
- ML CFA implicitly uses a strong prior with an exact zero loading
- BSEM uses an approximate zero loading using a zero-mean, small-variance prior for the loading:



BSEM can be used to specify approximate zeros for

- Cross-loadings
- Residual correlations
- Direct effects from covariates
- Group and time differences in intercepts and loadings

3.9 BSEM CFA vs ML CFA: Holzinger-Swineford 19 Vbles



CFA Factor Loading Pattern:

	Spatial	Verbal	Speed	Memory
visual	x	0	0	0
cubes	x	0	0	0
paper	x	0	0	0
flags	x	0	0	0
general	0	x	0	0
paragraf	0	x	0	0
sentence	0	x	0	0
wordc	0	x	0	0
wordm	0	x	0	0
addition	0	0	x	0
code	0	0	x	0
counting	0	0	x	0
straight	0	0	x	0
wordr	0	0	0	x
numberr	0	0	0	x
figurer	0	0	0	x
object	0	0	0	x
numberf	0	0	0	x
figurew	0	0	0	x

ML CFA Testing Results For Holzinger-Swineford Data For Grant-White (n =145) And Pasteur (n=156)

Model	χ^2	df	P-value	RMSEA	CFI
Grant-White					
CFA	216	146	0.000	0.057	0.930
EFA	110	101	0.248	0.025	0.991
Pasteur					
CFA	261	146	0.000	0.071	0.882
EFA	128	101	0.036	0.041	0.972

EFA has 6 (Grant-White) and 9 (Pasteur) significant cross-loadings

	Grant-White Factor Loading Patterns For EFA				Pasteur Factor Loading Pattern For EFA			
	Spatial	Verbal	Speed	Memory	Spatial	Verbal	Speed	Memory
visual	0.628*	0.065	0.091	0.085	0.580*	0.307*	-0.001	0.053
cubes	0.485*	0.050	0.007	-0.003	0.521*	0.027	-0.078	-0.059
paper	0.406*	0.107	0.084	0.083	0.484*	0.101	-0.016	-0.229*
flags	0.579*	0.160	0.013	0.026	0.687*	-0.051	0.067	0.101
general	0.042	0.752*	0.126	-0.051	-0.043	0.838*	0.042	-0.118
paragrap	0.021	0.804*	-0.056	0.098	0.026	0.800*	-0.006	0.069
sentence	-0.039	0.844*	0.085	-0.057	-0.045	0.911*	-0.054	-0.029
wordc	0.094	0.556*	0.197*	0.019	0.098	0.695*	0.008	0.083
wordm	0.004	0.852*	-0.074	0.069	0.143*	0.793*	0.029	-0.023
addition	-0.302*	0.029	0.824*	0.078	-0.247*	0.067	0.664*	0.026
code	0.012	0.050	0.479*	0.279*	0.004	0.262*	0.552*	0.082
counting	0.045	-0.159	0.826*	-0.014	0.073	-0.034	0.656*	-0.166
straight	0.346*	0.043	0.570*	-0.055	0.266*	-0.034	0.526*	-0.056
wordr	-0.024	0.117	-0.020	0.523*	-0.005	0.020	-0.039	0.726*
numberr	0.069	0.021	-0.026	0.515*	-0.026	-0.057	-0.057	0.604*
figurer	0.354*	-0.033	-0.077	0.515*	0.329*	0.042	0.168	0.403*
object	-0.195	0.045	0.154	0.685*	-0.123	-0.005	0.333*	0.469*
numberf	0.225	-0.127	0.246*	0.450*	-0.014	0.092	0.092	0.427*
figurew	0.069	0.099	0.058	0.365*	0.139	0.013	0.237*	0.291*

- CFA: Cross-loadings fixed at zero - the model is rejected
- A more realistic hypothesis: Small cross-loadings allowed
- Cross-loadings are not all identified in terms of ML
- Different alternative: Bayesian CFA with informative priors for cross-loadings: $\lambda \sim N(0, 0.01)$.

This means that 95% of the prior is in the range -0.2 to 0.2

VARIABLE: NAMES = id female grade agey agem school
! grade = 7/8
! school = 0/1 for Grant-White/Pasteur
visual cubes paper flags general paragraf sentence wordc
wordm addition code counting straight wordr numberr figurer
object numberf figurew deduct numeric problemr series arith-
met;
USEV = visual-figurew;
USEOBS = school eq 0;
DEFINE: STANDARDIZE visual-figurew;
ANALYSIS: ESTIMATOR = BAYES;
PROCESSORS = 2;
FBITER = 10000;

Input BSEM CFA 19 Items 4 Factors Crossloading Priors (Continued)

MODEL: spatial BY visual* cubes paper flags;
 verbal BY general* paragrap sentence wordc wordm;
 speed BY addition* code counting straight;
 memory BY wordr* numberr figurer object numberf figurew;
 spatial-memory@1;
 ! cross-loadings:
 spatial BY general-figurew*0 (a1-a15);
 verbal BY visual-flags*0 (b1-b4);
 verbal BY addition-figurew*0 (b5-b14);
 speed BY visual-wordm*0 (c1-c9);
 speed BY wordr-figurew*0 (c10-c15);
 memory BY visual-straight*0 (d1-d13);

MODEL PRIORS:

a1-d13 ~ N(0,.01);

OUTPUT: TECH1 TECH8 STDY;

PLOT: TYPE = PLOT2;

ML analysis

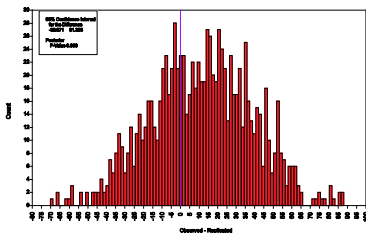
Model	χ^2	Df	P-value	RMSEA	CFI
Grant-White					
CFA	216	146	0.000	0.057	0.930
EFA	110	101	0.248	0.025	0.991
Pasteur					
CFA	261	146	0.000	0.071	0.882
EFA	128	101	0.036	0.041	0.972

Bayesian analysis

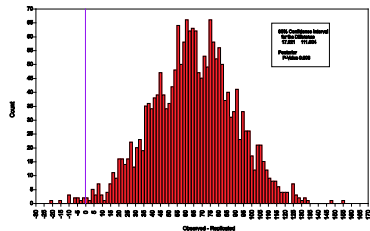
Model	Sample LRT	2.5% PP limit	97.5% PP limit	PP p-value
Grant-White				
CFA	219	12	112	0.006
CFA w/ cross-loadings	142	-39	61	0.361
Pasteur				
CFA	264	56	156	0.000
CFA w/ cross-loadings	156	-28	76	0.162

Bayesian Posterior Predictive Checking For The CFA Model For Grant-White

CFA with small cross-loadings
not rejected by Bayes PPC:
 $p = 0.361$



Conventional CFA model
rejected by Bayes PPC:
 $p = 0.006$



	Grant-White Factor Loadings Using Informative Priors				Pasteur Factor Loadings Using Informative Priors			
	Spatial	Verbal	Speed	Memory	Spatial	Verbal	Speed	Memory
visual	0.640*	0.012	0.050	0.047	0.633*	0.145	0.027	0.039
cubes	0.521*	-0.008	-0.010	-0.012	0.504*	-0.027	-0.041	-0.030
paper	0.456*	0.040	0.041	0.047	0.515*	0.018	-0.024	-0.118
flags	0.672*	0.046	-0.020	0.005	0.677*	-0.095	0.026	0.093
general	0.037	0.788*	0.049	-0.040	-0.056	0.856*	0.027	-0.084
paragrap	-0.001	0.837*	-0.053	0.030	0.015	0.801*	-0.011	0.050
sentence	-0.045	0.885*	0.021	-0.055	-0.063	0.925*	-0.032	-0.036
wordc	0.053	0.612*	0.096	0.029	0.055	0.694*	0.013	0.063
wordm	-0.012	0.886*	-0.086	0.020	0.092	0.803*	0.001	0.012
addition	-0.172*	0.030	0.795*	0.004	-0.147	-0.004	0.655*	0.010
code	-0.002	0.054	0.560*	0.130	-0.004	0.111	0.655*	0.049
counting	0.013	-0.092	0.828*	-0.049	0.025	-0.058	0.616*	-0.057
straight	0.189*	0.043	0.633*	-0.035	0.132	-0.067	0.558*	0.001
wordr	-0.040	0.044	-0.031	0.556*	-0.058	0.006	-0.090	0.731*
numberr	0.003	-0.004	-0.038	0.552*	0.006	-0.098	-0.106	0.634*
figurer	0.132	-0.024	-0.049	0.573*	0.156*	0.027	0.064	0.517*
object	-0.139	0.014	0.029	0.724*	-0.097	0.007	0.122	0.545*
numberf	0.099	-0.071	0.095	0.564*	-0.029	0.041	0.003	0.474*
figurew	0.012	0.045	0.007	0.445*	0.049	0.018	0.085	0.397*

Number of significant cross-loadings: 2 for Grant-White and 1 for Pasteur

Effects Of Using Different Variances For The Informative Priors Of The Cross-Loadings For The Holzinger-Swineford Data: Grant-White

Prior variance	95% cross-loading limit	PPP	Cross-loading (Posterior SD)	Factor corr. range
0.01	0.02	0.361	0.189 (.078)	0.443-0.557
0.02	0.28	0.441	0.248 (.096)	0.439-0.542
0.03	0.34	0.457	0.275 (.109)	0.432-0.530
0.04	0.39	0.455	0.292 (.120)	0.413-0.521
0.05	0.44	0.453	0.303 (.130)	0.404-0.513
0.06	0.48	0.447	0.309 (.139)	0.400-0.510
0.07	0.52	0.439	0.315 (.148)	0.395-0.508
0.08	0.55	0.439	0.319 (.156)	0.387-0.508
0.09	0.59	0.435	0.323 (.163)	0.378-0.506
0.10	0.62	0.427	0.327 (.171)	0.369-0.504

Summary of Analyses of Holzinger-Swineford 19-Variable Data

- Conventional, frequentist, CFA model rejected
- Bayesian CFA with informative cross-loadings not rejected
- The Bayesian approach uses an intermediate hypothesis:
 - Less strict than conventional CFA
 - Stricter than EFA, where the hypothesis only concerns the number of factors
 - Cross-loadings shrunken towards zero; acceptable degree of shrinkage monitored by PPP
- Bayes modification indices obtained by estimated cross-loadings
- Factor correlations: $EFA < BSEM < CFA$

- Target rotation: EFA rotation chosen to match zero target loadings using least-squares fitting
 - Similarities: Replaces mechanical rotation with judgement/hypotheses
 - Differences: For Target, specifying more than the necessary EFA restrictions does not affect fit and user-defined closeness to zero is replaced with least-squares fitting
- Results for Holzinger-Swineford data:
 - Results similar to EFA with 10 significant cross-loadings for Grant-White and 15 for Pasteur

Comparing BSEM And ESEM

- ESEM: Structural equation modeling with EFA measurement model (Asparouhov & Muthén (2009). Exploratory structural equation modeling. Structural Equation Modeling, 16, 397-438)
- Similarities: Both ESEM and BSEM can be used for measurement models in SEM
- Differences:
 - ESEM is EFA-oriented while BSEM is CFA-oriented
 - ESEM uses a mechanical rotation and the rotation is not based on information from other parts of the model
 - BSEM is applicable not only to measurement models

4. Analysis of Several Groups: Fixed vs Random Mode

- Group is fixed mode
 - Conventional multiple-group analysis
 - Multiple-group alignment optimization (Web Note 18): New in Version 7.1
 - Multiple-group BSEM with approximate invariance (Web Note 17): new in Version 7
- Group is random mode
 - Conventional two-level factor analysis (random intercepts)
 - Two-level analysis with random item parameters including random loadings: New in Version 7

4.1 Refresher on Multiple-Group Factor Analysis (Fixed Mode): 3 Different Degrees of Measurement Invariance

- 1 CONFIGURAL (invariant factor loading pattern)
- 2 METRIC (invariant factor loadings; "weak factorial invariance")
 - Needed in order to compare factor variances across groups
- 3 SCALAR (invariant factor loadings and intercepts/thresholds; "strong factorial invariance")
 - Needed in order to compare factor means across groups

These are automatically specified in Mplus Version 7.1 by 3 new options in the ANALYSIS command:

```
MODEL=CONFIGURAL METRIC SCALAR;
```

4.2 Input for Holzinger-Swineford 4-Group Analysis of 19 Variables: Gender \times School

VARIABLE: NAMES = id female grade agey agem school ! school =0/1 with Pasteur=1
visual cubes paper flags general paragrap sentence
wordc wordm addition code counting straight wordr
numberr figurer object numberf figurew deduct
numeric problemr series arithmet;
USEVARIABLES = visual-figurew group;
GROUPING = group (4);

DEFINE: IF (female EQ 0 AND school EQ 0)THEN group = 1; ! females in Grant-White
IF(female EQ 0 AND school EQ 1)THEN group = 2; ! females in Pasteur
IF(female EQ 1 AND school EQ 0)THEN group = 3; ! males in Grant-White
IF(female EQ 1 AND school EQ 1)THEN group = 4; ! males in Pasteur

ANALYSIS: ESTIMATOR = MLR;
MODEL = CONFIGURAL METRIC SCALAR;

MODEL: spatial BY visual-flags;
verbal BY general-wordm;
speed BY addition-straight;
memory BY wordr-figurew;

OUTPUT: TECH1;

Summary of Analysis

Number of groups	4
Number of observations	
Group G1 (1)	72
Group G2 (2)	82
Group G3 (3)	73
Group G4 (4)	74
Total sample size	301

Holz-Swineford: Model Fit Information - Invariance Testing

Model	Number of Parameters	Chi-square	Degrees of Freedom	P-value
Configural	252	837.003	584	0.0000
Metric	207	883.158	629	0.0000
Scalar	162	1102.935	674	0.0000

Models Compared	Chi-square	Degrees of Freedom	P-value
Metric against Configural	50.181	45	0.2755
Scalar against Configural	248.792	90	0.0000
Scalar against Metric	218.953	45	0.0000

Holz-Swineford: Model Fit for Configural Model

Chi-Square Contribution and P-Value From Each Group

(degrees of freedom = 146)

G1 (1)	211.266	0.000
G2 (2)	193.682	0.005
G3 (3)	193.128	0.005
G4 (4)	238.968	0.000

RMSEA (Root Mean Square Error Of Approximation)

Estimate	0.076	
90 Percent C.I.	0.064	0.087
Probability RMSEA \leq .05	0.000	

CFI/TLI

CFI	0.880
TLI	0.859

Chi-Square Test of Model Fit for the Baseline Model

Value	2784.560
Degrees of Freedom	684
P-Value	0.0000

SRMR (Standardized Root Mean Square Residual)

Value	0.086
-------	-------

4.3 Input for Binary Alcohol Items in NESARC: Two-Group Bi-Factor Modeling with Complex Survey Data

```
VARIABLE:  WEIGHT = swght;
           STRATIFICATION = stratum;
           CLUSTER = psu;
           IDVARIABLE = idnum;
           USEVARIABLES = ydep1-yabu4;
           USEOBSERVATIONS = male eq 1 AND native ne 1 AND asian ne 1 AND hisp ne 1;
           CATEGORICAL = ydep1-yabu4; ! Requires numerical integration
           CLASS = c(2);
           KNOWNCLASS = c (black = 0 1);
           !GROUPING = black (0=nonblack 1=black);
ANALYSIS:  TYPE = COMPLEX MIXTURE;
           ESTIMATOR = MLR;
           PROCESSORS = 8;
           MODEL = CONFIGURAL METRIC SCALAR;
MODEL:     %OVERALL%
           g BY ydep1-yabu4*;
           dep BY ydep1-ydep7*;
           abu BY yabu1-yabu4*;
           g-abu@1;
           g WITH dep-abu@0;
           ![g-abu@0];
```

NESARC: Model Fit Information - Invariance Testing

Model	Number of Parameters	Loglikelihood
Configural	69	-21665.014
Metric	50	-21681.061
Scalar	42	-21690.518

Models Compared	Chi-square	Degrees of Freedom	P-value
Metric against Configural	38.707	19	0.0048
Scalar against Configural	44.567	27	0.0181
Scalar against Metric	9.989	8	0.2658

4.4 Input for Multiple-Group ESEM of Aggressive-Disruptive Behavior

Input for ESEM analysis testing configural, metric, and scalar invariance for males and females using 13 continuous items measuring 3 factors:

```
VARIABLE:  GROUPING = gender (0=female 1=male);  
           USEVARIABLES = y301-y313;  
ANALYSIS:  ESTIMATOR = ML;  
           MODEL = CONFIGURAL METRIC SCALAR;  
           ROTATION = GEOMIN;  
MODEL:     f1-f3 BY y301-y313 (*1);  
OUTPUT:    TECH1 SAMPSTAT MODINDICES STANDARDIZED;
```

This is a simplified way of specifying the invariance models shown in ex5.27 of the Version 7 User's Guide.

4.5 Multiple-Group Factor Analysis: A New Method - Alignment Optimization

There is a need for a new approach to multiple-group factor analysis for many groups such as with country comparisons of achievement (PISA, TIMSS, PIRL) or cross-cultural studies (ISSP, ESS etc):

- Goal is to study measurement invariance and also group differences in factor means and variances
- Standard approach is confirmatory factor analysis with equality constraints
- The standard approach is too cumbersome to be practical for analysis of many groups where there can be a large number of non-invariant measurement parameters
- A radically different method is introduced in Mplus Version 7.1: Alignment optimization

Multiple-Group Alignment Optimization

Asparouhov & Muthén (2013). Multiple-group rotational alignment.
Web Note 18. New in Mplus Version 7.1

- 1 Step 1: Configural factor analysis - no across-group restrictions
 - 2 Step 2: Rotate according to a simplicity criterion
 - 3 Step 3: Adjust the factor means and factor variances in line with the rotation
- The simplicity criterion is optimized at a few large non-invariant parameters and many invariant parameters rather than many medium-sized non-invariant parameters (compare with EFA rotations using functions that aim for either large or small loadings, not mid-sized loadings)
 - Can be combined with BSEM allowing residual covariances with zero-mean, small-variance priors to improve model fit
 - Can be based on multiple-group BSEM with approximate measurement invariance instead of the usual configural model

CFA, ESEM, and BSEM with cross-loadings all fail in that too many instances of scalar measurement non-invariance are found: Factor means can not be compared across groups.

The problem is that these methods start with the scalar model of full invariance which is too far from the true model which has some large non-invariances and many ignorable non-invariances.

The alignment optimization method resolves this problem, making the factor means and variances comparable across groups and reducing the number of significant non-invariances.

4.6 Input for Nationalism & Patriotism Alignment in 34 Countries

```
DATA:          FILE = issp.txt;
VARIABLE:      NAMES = country v21 v22 v26 v29 v35;
                USEVARIABLES = v21-v35;
                MISSING = v21-v35 (0 8 9);
                CLASSES = c(34);
                !KNOWNCLASS = c(country = 1 2 4 6-8 10-22 24-28 30-33 36
                !40-43);
                KNOWNCLASS = c(country);
ANALYSIS:      TYPE = MIXTURE;
                ESTIMATOR = ML;
                ALIGNMENT = FREE;
MODEL:         %OVERALL%
                nat BY v21-v22;
                pat BY v26v35;
OUTPUT:        TECH1 TECH8 ALIGN;
```

- STANDARD ERROR COMPARISON INDICATES THAT THE FREE ALIGNMENT MODEL MAY BE POORLY IDENTIFIED. USING THE FIXED ALIGNMENT OPTION MAY RESOLVE THIS PROBLEM.

This leads to using FIXED Alignment:

ANALYSIS:

TYPE = MIXTURE;
ESTIMATOR = ML;
ALIGNMENT = FIXED(28);

Nationalism and Patriotism Example: Alignment Results

Approximate Measurement (Non-) Invariance by Group

Intercepts for Nationalism indicators (V21, V22) and Patriotism indicators (V26, V29, V35)

V21	1	2	3	4	5	6	7	8	9	10	11	12
	13	14	15	16	17	18	19	20	21	22	23	24
	25	26	27	28	29	30	31	32	33	34		
V22	(1)	2	3	(4)	5	(6)	7	8	(9)	10	11	12
	13	14	(15)	(16)	17	18	(19)	(20)	21	(22)	(23)	24
	(25)	26	27	28	(29)	30	31	(32)	33	34		
V26	1	2	3	4	5	6	7	8	9	10	11	12
	13	14	15	16	17	18	19	20	21	22	23	24
	25	26	27	28	29	30	31	32	33	34		
V29	(1)	2	3	(4)	(5)	6	7	(8)	(9)	10	11	12
	(13)	14	15	16	(17)	18	(19)	(20)	(21)	(22)	(23)	(24)
	(25)	26	27	28	29	(30)	31	32	33	(34)		
V35	(1)	(2)	3	(4)	5	6	7	(8)	(9)	(10)	11	12
	13	14	15	16	17	18	(19)	(20)	21	(22)	23	(24)
	25	26	(27)	(28)	(29)	(30)	31	32	(33)	34		

Nationalism and Patriotism Example: Factor Mean Comparisons (5% Significance Level)

Results for factor NAT

Ranking	Group	Value	Groups with significantly smaller factor mean
1	22	0.067	2 19 11 12 9 24 23 10 15 20 33 14 32 29 13 7 6 8 16 4 21 1 26 27 34 30 31 3 25 5
2	28	0.000	19 11 12 9 24 23 15 20 33 14 32 29 13 7 6 8 16 4 21 1 26 27 34 30 31 3 25 5 18 17
3	2	-0.284	6 16 4 21 1 26 27 34 31 3 25 5 18 17
4	19	-0.333	32 13 7 6 16 4 21 1 26 27 34 31 3 25 5 18 17
5	11	-0.344	33 32 13 7 6 16 4 21 1 26 27 34 31 3 25 5 18 17
6	12	-0.352	13 7 6 16 4 21 1 26 27 34 31 3 25 5 18 17
7	9	-0.357	7 6 16 4 21 1 26 27 34 31 3 25 5 18 17
8	24	-0.379	6 16 4 21 1 26 27 34 31 3 25 5 18 17
9	23	-0.388	13 7 6 16 4 21 1 26 27 34 31 3 25 5 18 17
10	10	-0.395	16 4 21 1 26 27 34 31 3 25 5 18 17
11	15	-0.396	13 7 6 16 4 21 1 26 27 34 31 3 25 5 18 17
12	20	-0.413	13 7 6 16 4 21 1 26 27 34 31 3 25 5 18 17

4.7 Alignment Optimization: Binary Math Items in 40 Countries (PISA)

- Items from the PISA (Program for International Student Assessment) survey of 2003
- A total of 9796 students from 40 countries
- Analyzed by Fox (2010). Bayesian Item Response Modeling
- A 40-group, one-factor model for eight mathematics test items
- 2-parameter probit IRT model that accommodates country measurement non-invariance for all difficulty (threshold) and discrimination (loading) parameters as well as country-specific factor means and variances

Input for PISA Alignment

```
DATA:          FILE = pisa2003.dat;
VARIABLE:     NAMES = cn y1-y8;
               CATEGORICAL = y1-y8; ! Requires Bayesian analysis
               USEVARIABLES = y1-y8;
               MISSING = y1-y8(9);
               CLASSES = c(40);
               KNOWNCLASS = c(cn = 1-40);
ANALYSIS:     TYPE = MIXTURE;
               ESTIMATOR = BAYES;
               PROCESSORS = 2;
               ALIGNMENT = FREE;
               THIN = 10;
               BITERATIONS = (5000);
MODEL:        %OVERALL%
               f BY y1-y8;
OUTPUT:       TECH1 TECH8 ALIGN;
PLOT:         TYPE = PLOT2;
```

Using zero-mean, small-variance priors.

- Single group analysis (2012 Psych Methods article):
 - Cross-loadings
 - Residual covariances
 - Direct effects in MIMIC
- Multiple-group analysis:
 - Configural and scalar analysis with cross-loadings and/or residual covariances
 - Approximate measurement invariance (Web Note 17)
 - BSEM-based alignment optimization (Web Note 18):
 - Residual covariances
 - Approximate measurement invariance

What does Bayes contribute?

- 1 Bayes with informative, zero-mean, small-variance priors for residual covariances can allow better configural fit - configural misfit in some groups is a common problem
- 2 Bayes with informative, zero-mean, small-variance priors for measurement parameter differences across groups (multiple-group BSEM) can allow better scalar fit
 - MG-BSEM as an alternative to alignment (finds non-invariance)
 - MG-BSEM-based alignment (advantageous for small samples?)
- 3 Bayes alignment can produce plausible values for the subjects' factor score values to be used in further analyses

Muthén & Asparouhov (2013). BSEM measurement invariance analysis. Web Note 17.

- Approximate measurement invariance across groups using zero-mean, small-variance informative priors for the group differences
- Produces "modification indices" by flagging non-invariant items as significantly deviating from average (ML-based MIs not available for categorical items)
- Freeing the non-invariant parameters gives proper "alignment", otherwise an alignment run is needed (BSEM-based alignment: ALIGNMENT = FREE(BSEM);)

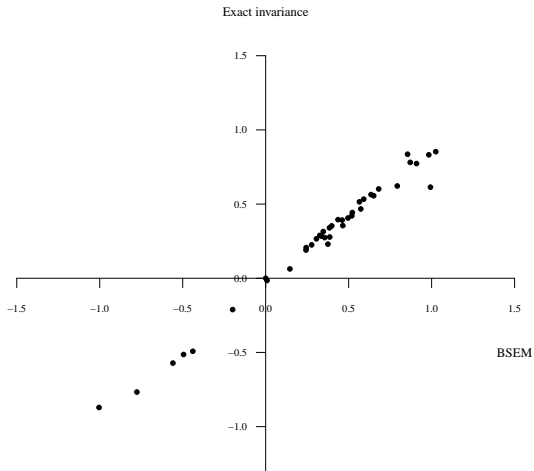
Multiple-Group BSEM: Non-Invariance Findings for PISA Items

Table : PISA countries with significant differences relative to the average across countries (prior variance = 0.10)

Item	Loading	Threshold
1	-	2, 12, 18, 22, 28, 39
2	15, 35	29, 38
3	15	23, 34, 35
4	-	12, 27, 40
5	3	7, 37
6	3, 33	5, 18, 25, 27, 37
7	-	9, 24, 27
8	24	-

Estimated Factor Means for 40 PISA Countries

Figure : Estimated factor means for 40 countries: Comparing BSEM analysis (X axis) with analysis imposing exact invariance (Y axis)



5. Switching to Random Mode: What Can Two-Level Factor Analysis Tell Us About Invariance?

Refresher on Two-Level Factor Analysis - 3 Major Types of Models:

- 1 Random intercepts: Different Within and Between factor structures (from factor analysis tradition)
- 2 Non-random intercepts: Same Within and Between factor structures and Between residual variances = 0 (used in IRT)
- 3 Random intercepts & random loadings (Bayesian analysis)

Two-Level Factor Analysis: Different Within and Between Factor Structures

Recall random effect ANOVA for individual i in cluster j ,

$$y_{ij} = \nu + y_{Bj} + y_{Wij}.$$

Two-level factor analysis generalizes this to

$$y_{ij} = \nu + \lambda_B f_{Bj} + \varepsilon_{Bj} + \lambda_W f_{Wij} + \varepsilon_{Wij}$$

with covariance structure $V(y_{ij}) = \Sigma_B + \Sigma_W$, where

$$\begin{aligned}\Sigma_B &= \Lambda_B \Psi_B \Lambda_B' + \Theta_B, \\ \Sigma_W &= \Lambda_W \Psi_W \Lambda_W' + \Theta_W.\end{aligned}$$

Random Intercept Two-Level Factor Analysis: Different Within and Between Factor Structures

The two-level factor analysis model

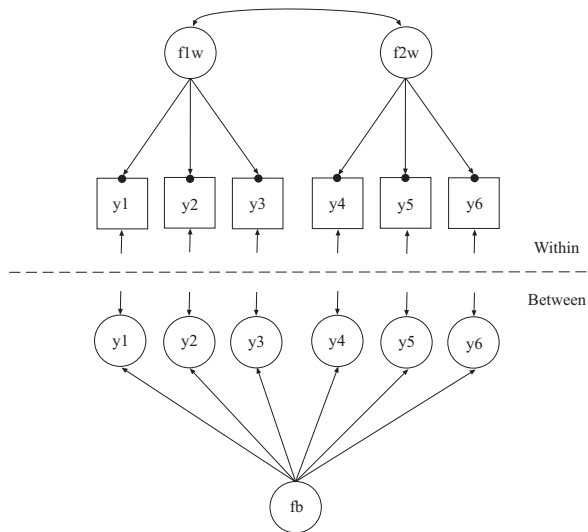
$$y_{ij} = \nu + \lambda_B f_{B_j} + \varepsilon_{B_j} + \lambda_W f_{W_{ij}} + \varepsilon_{W_{ij}}$$

can be viewed as a random intercept model:

$$\text{Level 1 : } y_{ij} = \nu_j + \lambda_W f_{W_{ij}} + \varepsilon_{W_{ij}},$$

$$\text{Level 2 : } \nu_j = \nu + \lambda_B f_{B_j} + \varepsilon_{B_j}.$$

Random Intercept Two-Level Factor Analysis in Figure Form



Connections Between Random Intercept Two-Level Factor Analysis, Conventional Two-Level IRT, and Measurement Invariance

- Random intercept two-level factor analysis:

$$\text{Level 1 : } y_{ij} = v_j + \lambda_W f_{W_{ij}} + \varepsilon_{W_{ij}},$$

$$\text{Level 2 : } v_j = v + \lambda_B f_{B_j} + \varepsilon_{B_j},$$

- Conventional two-level IRT:

If $\lambda_W = \lambda_B = \lambda$ and $V(\varepsilon_{B_j}) = 0$, then the above equations become

$$y_{ij} = v + \lambda f_{ij} + \varepsilon_{ij},$$

$$f_{ij} = f_{B_j} + f_{W_{ij}},$$

- The IRT model implies that we have measurement invariance across the clusters for both the intercepts and the loadings

5.1 Two-Level Analysis with Random Item Parameters

- De Jong, Steenkamp & Fox (2007). Relaxing measurement invariance in cross-national consumer research using a hierarchical IRT model. *Journal of Consumer Research*, 34, 260-278.
- Fox (2010). *Bayesian Item Response Modeling*. Springer
- Fox & Verhagen (2011). Random item effects modeling for cross-national survey data. In E. Davidov & P. Schmidt, and J. Billiet (Eds.), *Cross-cultural Analysis: Methods and Applications*
- Asparouhov & Muthén (2012). General random effect latent variable modeling: Random subjects, items, contexts, and parameters
- Bayesian estimation needed because random loadings with ML give rise to numerical integration with many dimensions

- Y_{ijk} - outcome for student i , in country j and item k

$$P(Y_{ijk} = 1) = \Phi(a_{jk}\theta_{ij} + b_{jk})$$

$$a_{jk} \sim N(a_k, \sigma_{a,k}), b_{jk} \sim N(b_k, \sigma_{b,k})$$

This is a 2-parameter probit IRT model where both discrimination (a) and difficulty (b) vary across country

- The θ ability factor is decomposed as

$$\theta_{ij} = \theta_j + \varepsilon_{ij}$$

- The mean and variance of the ability vary across country
- Model preserves common measurement scale while accommodating measurement non-invariance
- The ability for each country obtained by factor score estimation

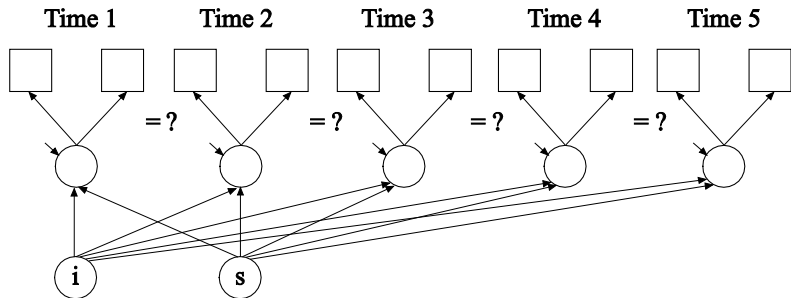
6. Longitudinal Analysis

- BSEM longitudinal approximate measurement invariance
 - Muthén & Asparouhov (2013). BSEM measurement invariance analysis. Web Note 17
- Intensive longitudinal data (many time points)
 - Individual differences factor analysis (TYPE=TWOLEVEL)
 - Cross-classified longitudinal analysis (TYPE=CROSSCLASSIFIED)

6.1 Advances In Multiple Indicator Growth Modeling

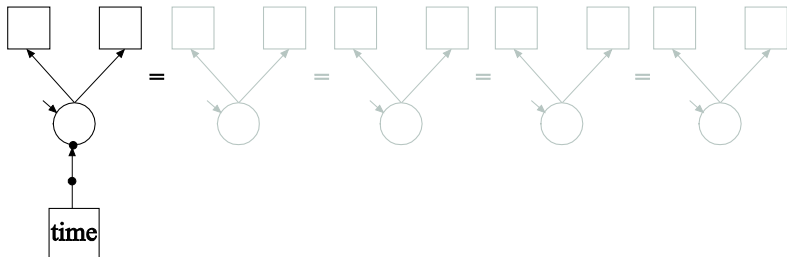
- An old dilemma
- Two new solutions

Categorical Items, Wide Format, Single-Level Approach



Single-level analysis with $p \times T = 2 \times 5 = 10$ variables, $T = 5$ factors.

- ML hard and impossible as T increases (numerical integration)
- WLSMV possible but hard when $p \times T$ increases and biased unless attrition is MCAR or multiple imputation is done first
- Bayes possible
- Searching for partial measurement invariance is cumbersome

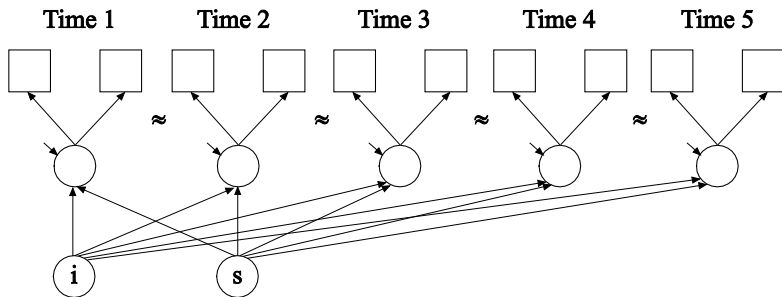


Two-level analysis with $p = 2$ variables, 1 within-factor, 2-between factors, **assuming full measurement invariance across time.**

- ML feasible
- WLSMV feasible (2-level WLSMV)
- Bayes feasible

- Both old approaches have problems
 - Wide, single-level approach easily gets significant non-invariance and needs many modifications
 - Long, two-level approach has to assume invariance
- New solution no. 1, suitable for small to medium number of time points
 - A new wide, single-level approach where time is a fixed mode
- New solution no. 2, suitable for medium to large number of time points
 - A new long, two-level approach where time is a random mode
 - No limit on the number of time points

New Solution No. 1: Wide Format, Single-Level Approach

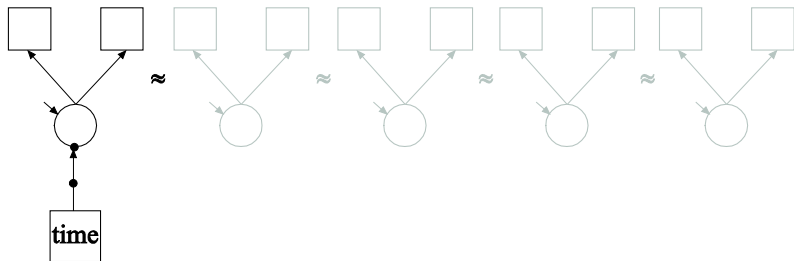


Single-level analysis with $p \times T = 2 \times 5 = 10$ variables, $T = 5$ factors.

- Bayes ("BSEM") using approximate measurement invariance, still identifying factor mean and variance differences across time

- New solution no. 2, time is a random mode
- A new long, two-level approach
 - Best of both worlds: Keeping the limited number of variables of the two-level approach without having to assume invariance

New Solution No. 2: Long Format, Two-Level (Cross-Classified) Approach



Two-level analysis with $p = 2$ variables.

- Bayes twolevel random approach with random measurement parameters and random factor means and variances using Type=Crossclassified: Clusters are time and person

6.2 BSEM for Aggressive-Disruptive Behavior in the Classroom

Randomized field experiment in Baltimore public schools with a classroom-based intervention aimed at reducing aggressive-disruptive behavior among elementary school students (Ialongo et al., 1999).

This analysis:

- Cohort 1
- 9 binary items at 8 time points, Grade 1 - Grade 7
- $n = 1174$

Aggressive-Disruptive Behavior in the Classroom: ML Versus BSEM For Binary Items

- Traditional ML analysis
 - 8 dimensions of integration
 - Computing time: 25:44 with
INTEGRATION=MONTECARLO(5000)
 - Increasing the number of time points makes ML impossible
- BSEM analysis
 - 156 parameters
 - Computing time: 4:01
 - Increasing the number of time points has relatively less impact

```
VARIABLE:    USEVARIABLES = stub1f-tease7s;  
             CATEGORICAL = stub1f-tease7s;  
             MISSING = ALL (999);  
DEFINE:      CUT stub1f-tease7s (1.5);  
ANALYSIS:    ESTIMATOR = BAYES;  
             PROCESSORS = 2;  
MODEL:       f1f by stub1f-tease1f* (lam11-lam19);  
             f1s by stub1s-tease1s* (lam21-lam29);  
             f2s by stub2s-tease2s* (lam31-lam39);  
             f3s by stub3s-tease3s* (lam41-lam49);  
             f4s by stub4s-tease4s* (lam51-lam59);  
             f5s by stub5s-tease5s* (lam61-lam69);  
             f6s by stub6s-tease6s* (lam71-lam79);  
             f7s by stub7s-tease7s* (lam81-lam89);  
             f1f@1;
```

```
[stub1f$1-tease1f$1] (tau11-tau19);  
[stub1s$1-tease1s$1] (tau21-tau29);  
[stub2s$1-tease2s$1] (tau31-tau39);  
[stub3s$1-tease3s$1] (tau41-tau49);  
[stub4s$1-tease4s$1] (tau51-tau59);  
[stub5s$1-tease5s$1] (tau61-tau69);  
[stub6s$1-tease6s$1] (tau71-tau79);  
[stub7s$1-tease7s$1] (tau81-tau89);  
[f1f-f7s@0];  
i s q | f1f@0 f1s@0.5 f2s@1.5 f3s@2.5 f4s@3.5  
f5s@4.5 f6s@5.5 f7s@6.5;  
q@0;
```

MODEL

PRIORS: DO(1,9) DIFF(lam1#-lam8#) ~ N(0,.01);
DO(1,9) DIFF(tau1#-tau8#) ~ N(0,.01);

OUTPUT: TECH1 TECH8;

Estimates For Aggressive-Disruptive Behavior

	Estimate	Posterior S.D.	One-Tailed P-Value	95% C.I.		
				Lower 2.5%	Upper 2.5%	
Means						
I	0.000	0.000	1.000	0.000	0.000	
S	0.238	0.068	0.000	0.108	0.366	*
Q	-0.022	0.011	0.023	-0.043	0.000	*
Variances						
I	9.258	2.076	0.000	6.766	14.259	*
S	0.258	0.068	0.000	0.169	0.411	*
Q	0.001	0.000	0.000	0.001	0.001	

Estimates For Aggressive-Disruptive Behavior, Continued

	Estimate	Posterior S.D.	One-Tailed P-Value	95% C.I.			
				Lower 2.5%	Upper 2.5%		
F1F	BY						
	STUB1F	0.428	0.048	0.000	0.338	0.522	*
	BKRULE1F	0.587	0.068	0.000	0.463	0.716	*
	HARMO1F	0.832	0.082	0.000	0.677	0.985	*
	BKTHIN1F	0.671	0.067	0.000	0.546	0.795	*
	YELL1F	0.508	0.055	0.000	0.405	0.609	*
	TAKEP1F	0.717	0.072	0.000	0.570	0.839	*
	FIGHT1F	0.480	0.052	0.000	0.385	0.579	*
	LIES1F	0.488	0.054	0.000	0.386	0.589	*
	TEASE1F	0.503	0.055	0.000	0.404	0.608	*
...							
F7S	BY						
	STUB7S	0.360	0.049	0.000	0.273	0.458	*
	BKRULE7S	0.512	0.068	0.000	0.392	0.654	*
	HARMO7S	0.555	0.074	0.000	0.425	0.716	*
	BKTHIN7S	0.459	0.063	0.000	0.344	0.581	*
	YELL7S	0.525	0.062	0.000	0.409	0.643	*
	TAKEP7S	0.500	0.069	0.000	0.372	0.634	*
	FIGHT7S	0.515	0.067	0.000	0.404	0.652	*
	LIES7S	0.520	0.070	0.000	0.392	0.653	*
	TEASE7S	0.495	0.064	0.000	0.378	0.626	*

Displaying Non-Invariant Items: Time Points With Significant Differences Compared To The Mean ($V = 0.01$)

Item	Loading	Threshold
stub	3	1, 2, 3, 6, 8
bkrule	-	5, 8
harmo	1, 8	2, 8
bkthin	1, 2, 3, 7, 8	2, 8
yell	2, 3, 6	-
takep	1, 2, 5	1, 2, 5
fight	1, 5	1, 4
lies	-	-
tease	-	1, 4, 8

6.3 Cross-Classified Longitudinal Analysis

- Both subject and time are random modes of variation (2 cluster variables)
- Observations nested within time and subject
- A large number of time points can be handled via Bayesian analysis
- A relatively small number of subjects is needed
- Mplus TYPE = CROSSCLASSIFIED
- Allows multiple indicator growth modeling with item parameters varying across time and subject (see UG ex9.27)

6.4 Two-Level Analysis with Random Loadings: Intensive Longitudinal Data

- Intensive longitudinal data (ILD): More and more longitudinal data are collected with very frequent observations using new tools for data collection such as palm pilots, smartphones etc.
- Ecological Momentary Assessment (EMA) involves repeated sampling of subjects' current behaviors and experiences in real time, in subjects' natural environments
- Experience Sampling Methods (ESM)
- Many time points, small number of subjects

- Walls & Schafer (2006). *Intensive Longitudinal Data*. New York: Oxford University Press
- Jahng, Wood & Trull (2008). Analysis of Affective Instability in Ecological Momentary Assessment: Indices Using Successive Difference and Group Comparison via Multilevel Modeling. *Psychological Methods*, 13, 354-375 (MSSD measure)
- Bolger & Laurenceau (2012). *Intensive Longitudinal Methods: An Introduction to Diary and Experience Sampling Research*. New York: Guilford Press
- Brose & Ram (2012). Within-Person Factor Analysis. In the new *Handbook of Research Methods for Studying Daily Life*

6.5 Individual Differences Factor Analysis: Two-Level Analysis with Random Factor Loadings

- Jahng S., Wood, P. K., & Trull, T. J., (2008). Analysis of Affective Instability in Ecological Momentary Assessment: Indices Using Successive Difference and Group Comparison via Multilevel Modeling. *Psychological Methods*, 13, 354-375
- An example of the growing amount of EMA data
- 84 outpatient subjects: 46 meeting borderline personality disorder (BPD) and 38 meeting MDD or DYS
- Each individual is measured several times a day for 4 weeks for total of about 100 assessments
- A mood factor for each individual is measured with 21 self-rated continuous items
- The research question is if the BPD group demonstrates more temporal negative mood instability than the MDD/DYS group

Individual Differences Factor Analysis (IDFA)

- This data set is suitable for checking if a measurement instrument is interpreted the same way by different individuals. Some individuals responses may be more correlated for some items, i.e., the correlation matrix could be different for different individuals
- Suppose that one individual always answers item 1 and 2 the same way and a second individual doesn't. We need separate factor analysis models for the two individuals, that is, individual-specific factor loadings
- If the within-level correlation matrix varies across individuals that means that the loadings are individual-specific
- Should factor loadings be individually specific in general? This cannot be determined in cross-sectional studies, only in longitudinal studies with multiple assessments
- IDFA uses TYPE=TWOLEVEL where cluster = individual with many assessments per cluster

- Large across-time variance of the mood factor is considered a core feature of BPD that distinguishes this disorder from other disorders like depressive disorders.
- The individual-specific factor variance is the most important feature in this study
- The individual-specific factor variance is confounded with individual-specific factor loadings
- How to separate the two? Answer: **Using IDFA with a factor model for the random factor loadings**
- Asparouhov & Muthén, B. (2012). General Random Effect Latent Variable Modeling: Random Subjects, Items, Contexts, and Parameters

Individual Differences Factor Analysis

Let Y_{rit} be item r , for individual i , at assessment t , measuring factor f_{it} .
The model is given by

$$Y_{rit} = \mu_r + \zeta_{ri} + \lambda_{ri} f_{it} + \varepsilon_{rit},$$

$$\lambda_{ri} = \lambda_r + \lambda_r \sigma_i + \varepsilon_{ri},$$

$$f_{it} = f_i + \xi_{it},$$

$$f_i = \beta_1 X_i + \zeta_{1i},$$

$$\sigma_i = \beta_2 f_i + \zeta_{2i}.$$

where the covariate X_i is a dummy variable for BPD versus MDD.
IDFA has individual-specific: item intercepts, item loadings, factor mean, factor variance.

Mplus Input for Individual Differences Factor Analysis of the 21 Mood Items

```
ANALYSIS:          TYPE = TWOLEVEL RANDOM;
                   ESTIMATOR = BAYES;

MODEL:             %WITHIN%
                   lam1-lam21 | f BY jittery-scornful;
                   f@1;
                   %BETWEEN%
                   lam1-lam21 jittery-scornful;
                   [lam1-lam21*1] (lambda1-lambda21);
                   sigma BY lam1-lam21*1 (lambda1-lambda21);
                   f ON x (b1); f;
                   sigma ON f (b2); sigma;

MODEL
CONSTRAINT:       NEW(indirect);
                   indirect = b1*b2;
```

Individual Differences Factor Analysis Results

All variance components are significant. Percent Loading Invariance = the percentage of the variation of the loadings across subjects that is explained by factor variance variation.

item	Res Var	Mean	Var of Mean	Loading	Var of Loading	Percent Loading Invariance
Item 1	0.444	1.505	0.287	0.261	0.045	0.29
Item 2	0.628	1.524	0.482	0.377	0.080	0.32
Item 3	0.331	1.209	0.057	0.556	0.025	0.77
Item 4	0.343	1.301	0.097	0.553	0.030	0.73
Item 5	0.304	1.094	0.017	0.483	0.053	0.54

Item 1: Jittery (less invariant). Item 3: Upset (more invariant)

- Clear evidence that measurement items are not interpreted the same way by different individuals and thus individual-specific adjustments are needed to the measurement model to properly evaluate the underlying factors: IDFA model
- IDFA model clearly separates factor variance variation from the factor loadings variation
- The mood factor mean is significantly higher for the BPD group
- The mood factor variance is significantly higher for subjects with higher mood factor values
- Indirect effect: The mood factor variance is significantly higher for the BPD group

Mean Square Successive Difference (MSSD) Computed from the Factor Scores

- Observed score MSSD:

$$MSSD = \frac{1}{T-1} \sum_{t=1}^{T-1} (y_{t+1} - y_t)^2 \quad (4)$$

- The same formula can be applied to the factor scores obtained by Bayes plausible value draws from the estimated IDFA model
- 5 plausible values is sufficient to estimate group characteristics such as the mean MSSD in the BPD group (2.117) and the MDD group (1.676). Can also be broken down into within-day and between-day MSSD (Jahng, Wood, Trull, 2008).

Current and Future Methods for Intensive Longitudinal Data

- Dynamic Factor Analysis (see, e.g., Zhang, Hamaker, Nesselroade, 2008 in SEM and references therein)
- Time series analysis (see Hamaker talk at the 4th Dutch Mplus Users' Group meeting via the Mplus home page)
- Mplus latent variable time series developments (see Asparouhov talk at the same meeting)

7. Mixture Modeling: Overview

- Zero-effect class CACE modeling
- 3-step mixture modeling: Analyze-classify-analyze approaches to investigate covariates and distal outcomes
 - LCA
 - Regression mixture analysis
 - GMM
 - LTA
- Latent transition analysis (LTA)
 - Introductory examples
 - New Mplus output
 - Covariates influencing transition probabilities
 - Probability parameterization useful for Mover-Stayer LTA
 - LTA extensions

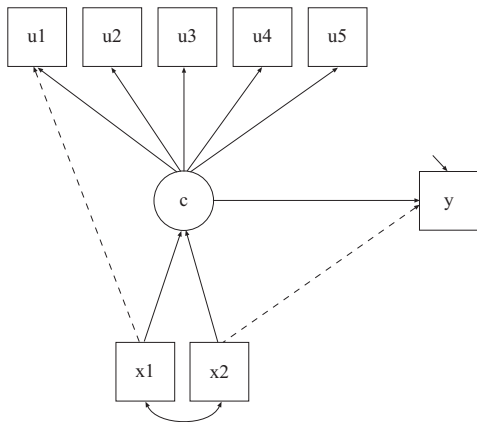
Complier-average causal effect (CACE) modeling is popular:

- Overall treatment effect often insignificant while the effect in the complier class is significant (e.g. JOBS data)
- But the modeling has weaknesses, for example, all compliers are assumed to benefit equally from the treatment
- New solution: Add a complier class with no treatment effect
- Fits JOBS data better and shows a sizeable group who don't benefit, giving a different treatment effect

Sobel & Muthén (2012). Compliance mixture modelling with a zero effect complier class and missing data. *Biometrics*, 68, 1037-1045

7.2 3-Step Mixture Modeling

1-step analysis versus 3-step (analyze-classify-analyze) latent class analysis



1-Step vs 3-Step: A Hypothetical Genetic Example

Substantive question: Should the latent classes be defined by the indicators alone or also by covariates and distal outcomes (antecedents and consequences)?

- Example: Study of genotypes (x variables) influencing phenotypes (y variables)
- Phenotypes may be observed indicators of mental illness such as DSM criteria. The interest is in finding latent classes of subjects and then trying to see if certain genotype variables influence class membership
- Possible objection to 1-step: If the genotypes are part of deciding the latent classes, the assessment of the strength of relationship is compromised
- 3-step: Determine the latent classes based on only phenotype information. Then classify subjects. Then relate the classification to the genotypes

- Latent class models should be subjected to both statistical and substantive checking (Muthén, 2003 in Psychological Methods)
- Substantive checking can be done by relating latent classes to antecedents and consequences (covariates and distal outcomes)
- The 3-step approach is a useful tool for this

The Old 3-Step Approach

- 1 Estimate the LCA model
- 2 Determine each subject's most likely class membership
- 3 Relate the most likely class variable to other variables

The old 3-step approach is problematic: Unless the classification is very good (high entropy), this gives biased estimates and biased standard errors for the relationships with other variables.

The LCA Provides Information About the Classification Quality

Average Latent Class Probabilities for Most Likely Class Membership (Row) by Latent Class (Column)

	1	2	3
1	0.839	0.066	0.095
2	0.053	0.845	0.102
3	0.125	0.107	0.768

The New 3-Step Approach

- New Method in Mplus Version 7: 3-Step approach correcting for classification error
 - ① Estimate the LCA model
 - ② Create a nominal most likely class variable N
 - ③ Use a mixture model for N , C and X , where N is a C indicator with measurement error rates prefixed at the misclassification rate of N estimated in the step 1 LCA analysis
- Bolck, Croon, & Hagenaars (2004) Estimating latent structure models with categorical variables: One-step versus three-step estimators. *Political Analysis*, 12, 3-27.
- Vermunt (2010). Latent Class Modeling with Covariates: Two improved three-step approaches. *Political Analysis*, 18, 450-469
- Asparouhov & Muthén (2012). Auxiliary variables in mixture modeling: A 3-step approach using Mplus. Mplus Web Note 15.

Average Latent Class Probabilities for Most Likely
Class Membership (Row) by Latent Class (Column)

	1	2	3
1	0.839	0.066	0.095
2	0.053	0.845	0.102
3	0.125	0.107	0.768

$$\log(0.839/0.095) = 2.178$$

$$\log(0.066/0.095) = -0.364$$

$$\log(0.053/0.102) = -0.654$$

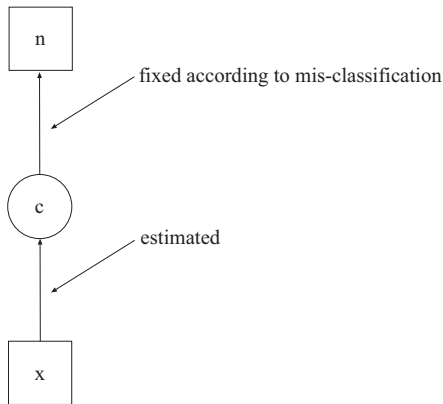
$$\log(0.845/0.102) = 2.114$$

$$\log(0.125/0.768) = -1.815$$

$$\log(0.107/0.768) = -1.970$$

Step 3 Regression on a Covariate

- n: Most likely class membership from Step 2 (nominal variable)
- c: Latent class variable
- x: Covariate



Input File for Step 3 in the 3-Step Estimation

```
VARIABLE:  NAMES = u1-u5 x p1-p3 n;  
           USEVARIABLES = x n;  
           CLASSES = c(3);  
           NOMINAL = n;  
DATA:      FILE = man3step2.dat;  
ANALYSIS:  TYPE = MIXTURE; STARTS = 0;  
MODEL:     %OVERALL%  
           c ON x;  
           %c#1%  
           [n#1@2.178];  
           [n#2@-0.364];  
           %c#2%  
           [n#1@-0.654];  
           [n#2@2.114];  
           %c#3%  
           [n#1@-1.815];
```

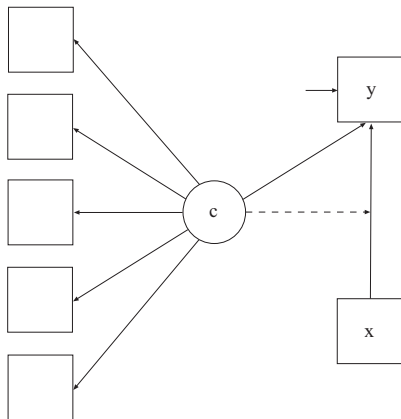
```
VARIABLE:  NAMES = u1-u5 x;  
           CATEGORICAL = u1-u5;  
           CLASSES = c(3);  
           AUXILIARY = x(R3STEP);  
DATA:      FILE = 3step.dat;  
ANALYSIS:  TYPE = MIXTURE;  
MODEL:     !no model is needed, LCA is default
```

```
VARIABLE:  NAMES = u1-u5 y;  
           CATEGORICAL = u1-u5;  
           CLASSES = c(3);  
           AUXILIARY = y(DU3STEP);  
DATA:      FILE = 3step.dat;  
ANALYSIS:  TYPE = MIXTURE;  
MODEL:     !no model is needed, LCA is default
```

A Second Look at Distal 3-Step

- In some examples the Asparouhov-Muthén distal 3-step method in Mplus Web Note 15 leads to changes in latent class formation between Step 1 and Step 3 - warning given in Mplus Version 7.1
- Lanza et al. (2013) in the SEM journal propose a different distal 3-step method that avoids changes in class formation. Included in Mplus Version 7.1
- Future research needed to evaluate which method, including Most Likely Class and Pseudo-class, is least sensitive to violations of assumptions such as no direct effects

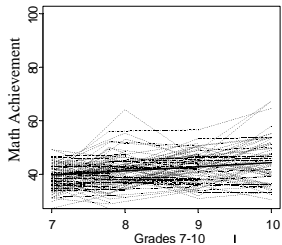
Manual 3-Step Mixture Modeling For Special Models: A Regression Mixture Example



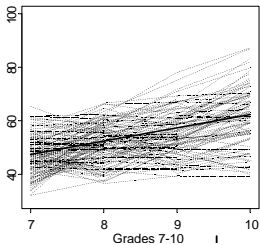
Growth Mixture Modeling: Math Achievement Trajectory Classes and High School Dropout.

An Example of Substantive Checking via Predictive Validity

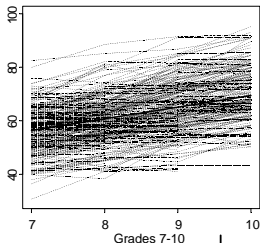
Poor Development: 33%



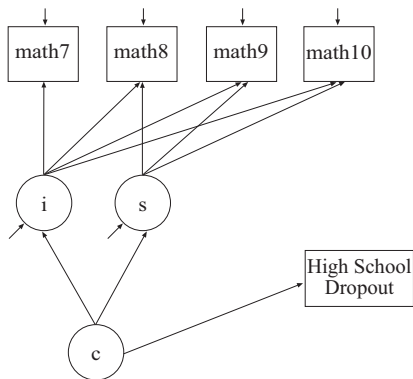
Moderate Development: 25%



Good Development: 43%



LSAY Math Achievement Trajectory Class Modeling and High School Dropout

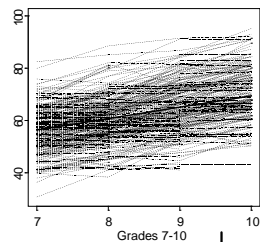
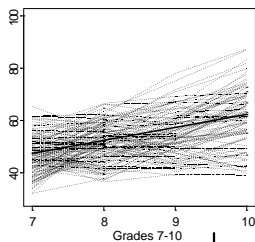
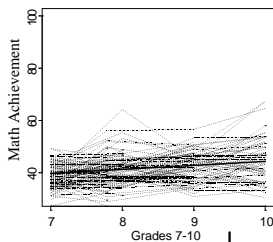


Growth Mixture Modeling: Math Achievement Trajectory Classes and High School Dropout

Poor Development: 20%

Moderate Development: 28%

Good Development: 52%



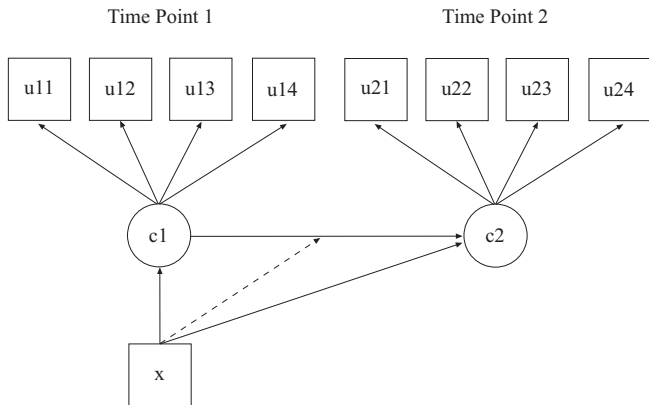
Dropout:

69%

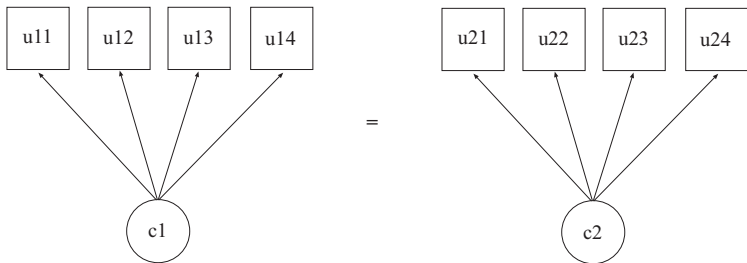
8%

1%

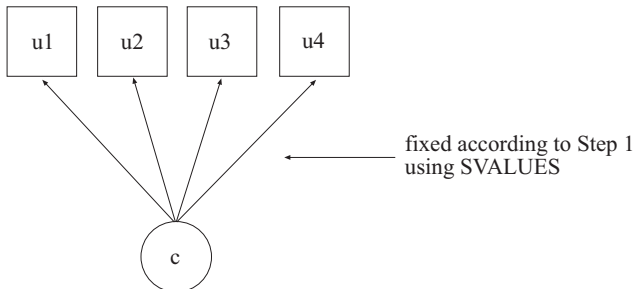
3-Step Latent Transition Analysis



LTA: Step 1

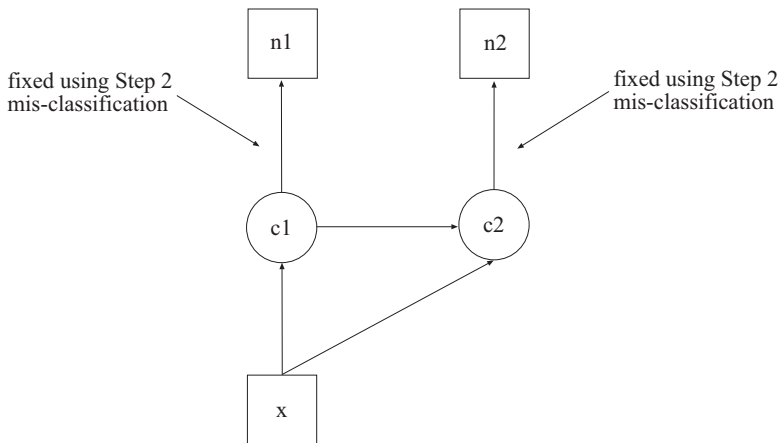


For each time point:

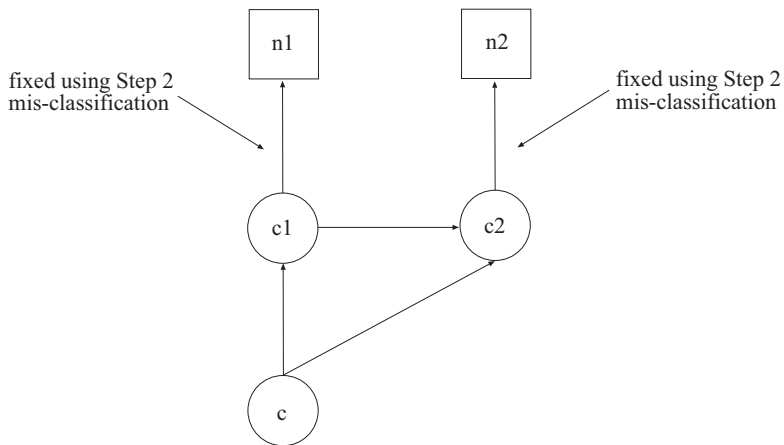


Save most likely class

LTA: Step 3



3-Step Mover-Stayer LTA



Feingold et al. (2013). New Approaches for Examining Associations with Latent Categorical Variables: Applications to Substance Abuse and Aggression. *Psychology of Addictive Behaviors*, DOI: 10.1037/a0031487

The 2013 Modern Modeling Methods (M3) conference had a symposium on stepwise mixture methods.

7.3 Latent Transition Analysis Developments

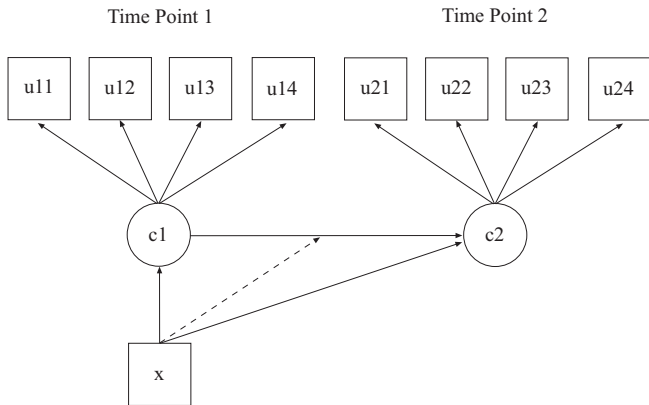
New developments in Version 7:

- TECH15 output with conditional class probabilities useful for studying transition probabilities varying as a function of an observed binary or nominal covariate such as treatment/control, ethnicity, or a latent class covariate
- LTA transition probability calculator for continuous covariates
- Probability parameterization to simplify input for Mover-Stayer LTA and other models with restrictions on the transition probabilities
- New User's Guide examples
 - 8.13: LTA for two time points with a binary covariate influencing the latent transition probabilities
 - 8.14: LTA for two time points with a continuous covariate influencing the latent transition probabilities
 - 8.15: Mover-stayer LTA for three time points using a probability parameterization

Latent Transition Analysis

Transition
Probabilities

		c2	
		1	2
c1	1	0.8	0.2
	2	0.4	0.6



LTA Example 1: Stage-Sequential Development in Reading Using ECLS-K Data

Kaplan (2008). An overview of Markov chain methods for the study of stage-sequential developmental processes. *Developmental Psychology*, 44, 457-467.

- Early Childhood Longitudinal Study - Kindergarten cohort
- Four time points: Kindergarten Fall, Spring and Grade 1 Fall, Spring; $n = 3,575$
- Five dichotomous proficiency scores: Letter recognition, beginning sounds, ending letter sounds, sight words, words in context
- LCA suggests 3 classes: Low alphabet knowledge (LAK), early word reading (EWR), and early reading comprehension (ERC)
- Binary poverty index

- Three latent classes:
 - Class 1: Low alphabet knowledge (LAK)
 - Class 2: Early word reading (EWR)
 - Class 3: Early reading comprehension (ERC)

The ECLS-K LTA model has the special feature of specifying no decline in knowledge as zero transition probabilities. For example, transition from Kindergarten Fall to Spring:

LATENT TRANSITION PROBABILITIES
BASED ON THE ESTIMATED MODEL

c1 classes (rows) by c2 classes (columns)

	1	2	3
1	0.329	0.655	0.017
2	0.000	0.646	0.354
3	0.000	0.000	1.000

LTA Example 1: ECLS-K.

Transition Tables for the Binary Covariate Poverty

		Poverty = 0			Poverty = 1		
		c2			c2		
		1	2	3	1	2	3
c1	1	0.252	0.732	0.017	0.545	0.442	0.013
	2	0.000	0.647	0.353	0.000	0.620	0.380
	3	0.000	0.000	1.000	0.000	0.000	1.000

LTA Example 2: Mover-Stayer LTA Modeling of Peer Victimization During Middle School

Nylund (2007) Doctoral dissertation: Latent Transition Analysis: Modeling Extensions and an Application to Peer Victimization

- Student's self-reported peer victimization in Grade 6, 7, and 8
- Low SES, ethnically diverse public middle schools in the Los Angeles area (11% Caucasian, 17% Black, 48 % Latino, 12% Asian)
- $n = 2045$
- 6 binary items: Picked on, laughed at, called bad names, hit and pushed around, gossiped about, things taken or messed up (Neary & Joseph, 1994 Peer Victimization Scale)

LTA Example 2: Mover-Stayer Model

- Class 1: Victimized (G6-G8: 19%, 10%, 8%)
- Class 2: Sometimes victimized (G6-G8: 34%, 27%, 21%)
- Class 3: Non-victimized (G6-G8: 47%, 63%, 71%)

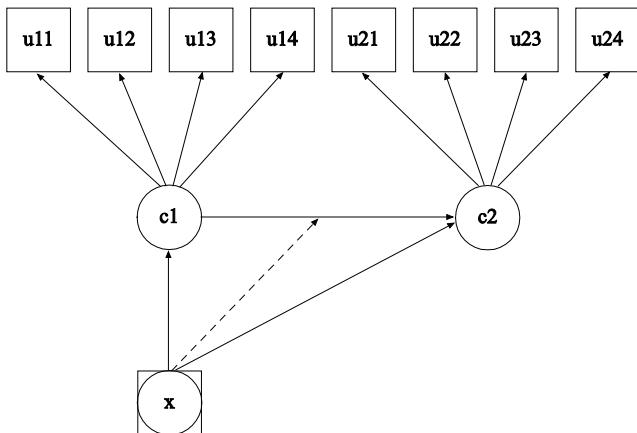
Movers (60%)									
		c2 (Grade 7)					c3 (Grade 8)		
		0.29	0.45	0.26			0.23	0.59	0.18
c1 (Grade 6)		0.06	0.44	0.51	c2 (Grade 7)		0.04	0.47	0.49
		0.04	0.46	0.55			0.06	0.17	0.77

Stayers (40%)									
		c2 (Grade 7)					c3 (Grade 8)		
		1	0	0			1	0	0
c1 (Grade 6)		0	1	0	c2 (Grade 7)		0	1	0
		0	0	1			0	0	1

- TECH15 output with conditional class probabilities useful for studying transition probabilities with an observed binary covariate such as treatment/control or a latent class covariate
- LTA transition probability calculator for continuous covariates
- Probability parameterization to simplify input for Mover-Stayer LTA and other models with restrictions on the transition probabilities

Latent Transition Probabilities Influenced by a Categorical Covariate: Using TECH15

- Covariate treated as Knownclass
- New feature in Version 7: TECH 15 output - conditional probabilities for "c ON c"



$$P(CX=1)=0.808$$

$$P(CX=2)=0.192$$

$$P(C1=1|CX=1)=0.617$$

$$P(C1=2|CX=1)=0.351$$

$$P(C1=3|CX=1)=0.032$$

$$P(C1=1|CX=2)=0.872$$

$$P(C1=2|CX=2)=0.123$$

$$P(C1=3|CX=2)=0.005$$

$$P(C2=1|CX=1,C1=1)=0.252$$

$$P(C2=2|CX=1,C1=1)=0.732$$

$$P(C2=3|CX=1,C1=1)=0.017$$

$$P(C2=1|CX=1,C1=2)=0.000$$

$$P(C2=2|CX=1,C1=2)=0.647$$

$$P(C2=3|CX=1,C1=2)=0.353$$

Latent Transition Probabilities Influenced By a Continuous Covariate

- Muthén & Asparouhov (2011). LTA in Mplus: Transition probabilities influenced by covariates. Mplus Web Notes: No. 13. July 27, 2011. www.statmodel.com
- New feature in Version 7: The LTA calculator

LTA Calculator Applied to Poverty

The screenshot displays the Mplus software interface. The main window title is "Mplus - [8-14 final lta first and second tech15 poverty as x 6-23.out]". The menu bar includes File, Edit, View, Mplus, Plot, Window, and Help. The Mplus menu is open, showing options: Language Generator, Run Mplus (Alt+R), and LTA calculator. Below the menu, the text "Mplus DEVELOPMENT" and "MUTHEN & MUTHEN" is visible, along with the date "08/14/2012".

A dialog box titled "Covariate values for adjusted estimated means" is open. It contains the following elements:

- Section: "Covariate sets for adjusted estimated means"
- Text: "Choose one of the following: name a new covariate set, display a covariate set, remove a covariate set."
- Buttons: "Name covariate set", "Display covariate set", "Remove covariate set"
- Fields: "pov" (selected in both "Display covariate set" and "Remove covariate set" dropdowns)
- Text: "Number of graphs for each group: 1"
- Section: "Covariate values"
- Table with columns "Variable" and "POV"
- Dialog box "Set value" is open over the table, showing "Value setting for: POV" and radio buttons for "Use sample mean" (selected) and "Use 0".
- Buttons: "OK", "Cancel" (at the bottom of the main dialog)

- New feature in Mplus Version 7
- LTA models that do not have continuous x 's can be more conveniently specified using
PARAMETERIZATION=PROBABILITY
to reflect hypotheses expressed in terms of probabilities
- Useful for Mover-Stayer LTA models

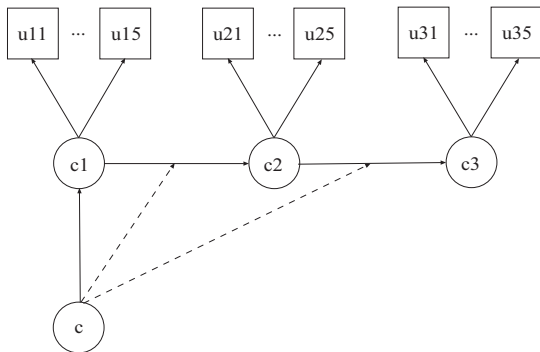
Probability parameterization for CLASSES = c1(3) c2(3):

		c2		
		1	2	3
c1	1	p11	p12	0
	2	p21	p22	0
	3	p31	p32	0

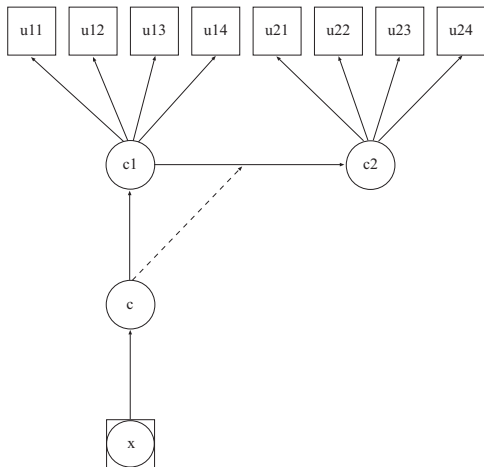
where the probabilities in each row add to 1 and the last c2 class is not mentioned. The p parameters are referred to using ON. The latent class variable c1 which is the predictor has probability parameters [c1#1 c1#2], whereas "intercept" parameters are not included for c2.

A transition probability can be conveniently fixed at 1 or 0 by using the p parameters.

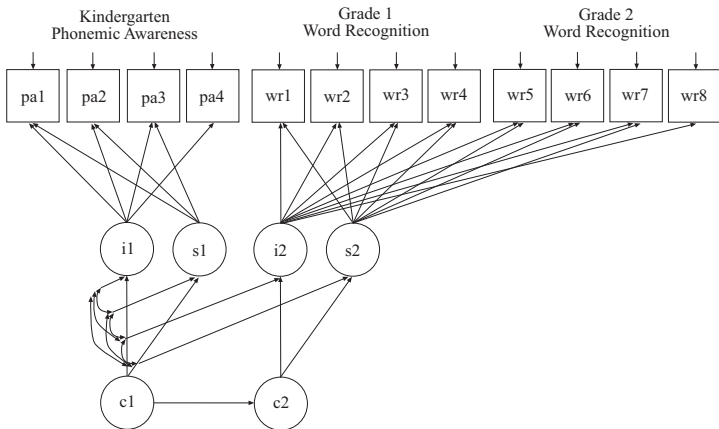
Mover-Stayer LTA in Probability Parameterization: UG ex 8.15



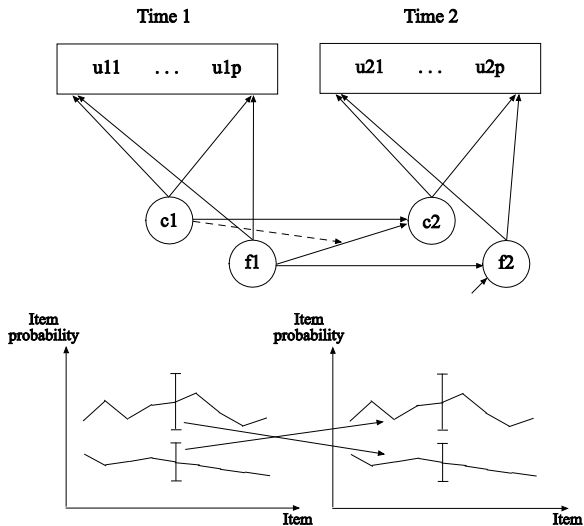
Mover-Stayer LTA in Probability Parameterization: Predicting Mover-Stayer Class Membership From A Nominal Covariate



Latent Transition Analysis Extensions: Latent Variables as LTA Indicators



Latent Transition Analysis Extensions: Factor Mixture Latent Transition Analysis (Muthén, 2006)



8. Multilevel Modeling

- Within-cluster multiple-group modeling
- Advantages of Bayesian analysis
- Meta analysis (2-level random)
- 3-level analysis
- Cross-classified analysis
- 3-level and cross-classified multiple imputation
- Applications to Item Response Theory modeling

8.1 Within-Cluster Multiple-Group Modeling

Asparouhov & Muthén (2012). Multiple group multilevel analysis. Web note 16.

Two-level SEM with multiple groups:

- Group as level-2 variable (e.g. public vs private schools) is straightforward - independent groups
- Group as level-1 variable (e.g. gender within schools) needs special treatment - groups are correlated due to common level-2 units
- New approach:
 - Mixture solution - level-1 groups are latent classes
 - Allows different level-1 groups to have different cluster effects (teacher effect different for males and females)

8.2 Advantages of Bayesian Multilevel Analysis

With carefully chosen priors, Bayes allows a smaller number of level 2 or level 3 units. See, e.g.,

Muthén (2010). Bayesian analysis in Mplus: A brief introduction. Technical Report. Version 3.

8.3 Meta Analysis (See Also Topic 9)

- Meta-analysis pools information from several studies designed to address the same scientific question
- Data frequently are in the form of summary statistics from each study, such as effect measures, means, (log) odds ratios, relative risks, z-transformed correlations, and the associated sampling variances
- A normal model for the summary statistic y_j in study j assumes $y_j \sim N(\theta_j, \sigma_j^2)$,
where σ_j^2 is assumed known, estimated from data
- A random-effects model specifies $\theta_j \sim N(\mu, \tau^2)$
- A Bayesian model adds priors such as $\mu \sim N(0, 1000)$, $\tau^2 \sim U(0, 1000)$

- Heterogeneous variances can be handled by random slopes as in Mplus Web Note # 3. See also Users Guide ex 3.9 (random coefficient regression)
- A similar approach is used in Cheung (2008). A model for integrating fixed-, random-, and mixed-effects meta-analyses into structural equation modeling. Psychological Methods, 13, 182-202
$$y_j = \theta_j + \varepsilon_j; \varepsilon_j \sim N(0, \sigma_j^2)$$
Dividing by σ_j ,
$$y_j^* = 0 + \theta_j * x_j + \varepsilon_j; \varepsilon_j \sim N(0, 1)$$
where $y_j^* = y_j/\sigma_j, x_j = 1/\sigma_j$ and θ_j is a random slope $\sim N(\mu, \tau^2)$
- Random slope approach using ML or Bayes
TYPE=TWOLEVEL RANDOM
- See video and handout for Topic 9 of 6/1/11, slides 150-169

8.4 Three-Level Analysis

Continuous outcomes: ML and Bayesian estimation

Categorical outcomes: Bayesian estimation (Bayes uses probit)

Count and nominal outcomes: Not yet available

Types Of Observed Variables In 3-Level Analysis

Each Y variable is decomposed as

$$Y_{ijk} = Y_{1ijk} + Y_{2jk} + Y_{3k},$$

where Y_{1ijk} , Y_{2jk} , and Y_{3k} are components of Y_{ijk} on levels 1, 2, and 3. Here, Y_{2jk} , and Y_{3k} may be seen as random intercepts on respective levels, and Y_{1ijk} as a residual

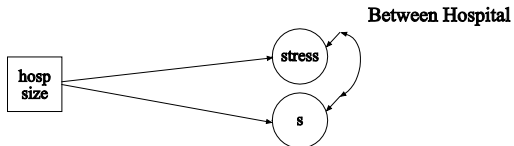
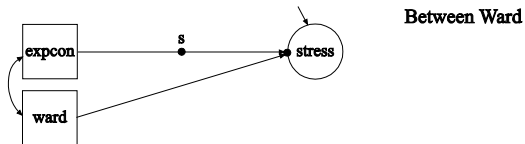
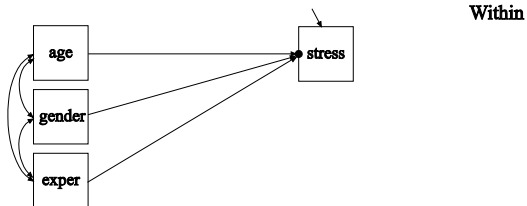
- Some variables may not have variation over all levels. To avoid variances that are near zero which cause convergence problems specify/restrict the variation level
- WITHIN=Y, has variation on level 1, so Y_{2jk} and Y_{3k} are not in the model
- WITHIN=(level2) Y, has variation on level 1 and level 2
- WITHIN=(level3) Y, has variation on level 1 and level 3
- BETWEEN= Y, has variation on level 2 and level 3
- BETWEEN=(level2) Y, has variation on level 2
- BETWEEN=(level3) Y, has variation on level 3

- Type 1: Defined on the level 1
%WITHIN%
 $s \mid y \text{ ON } x;$
The random slope s has variance on level 2 and level 3
- Type 2: Defined on the level 2
%BETWEEN level2%
 $s \mid y \text{ ON } x;$
The random slope s has variance on level 3 only
- The dependent variable can be an observed Y or a factor. The covariate X should be specified as WITHIN= for type 1 or BETWEEN=(level2) for type 2, i.e., no variation beyond the level it is used at

Source: Hox (2010). Multilevel Analysis. Hypothetical data discussed in Section 2.4.3

- Study of stress in hospitals
- Reports from nurses working in wards nested within hospitals
- In each of 25 hospitals, 4 wards are selected and randomly assigned to experimental or control conditions (cluster-randomized trial)
- 10 nurses from each ward are given a test that measures job-related stress
- Covariates are age, experience, gender, type of ward (0=general care, 1=special care), hospital size (0=small, 1=medium, 2=large)
- Research question: Is the experimental effect different in different hospitals? - Random slope varying on level 3

3-Level Regression Example: Nurses Data



Input For Nurses Data

TITLE: Nurses data from Hox (2010)
DATA: FILE = nurses.dat;
VARIABLE: NAMES = hospital ward wardid nurse age gender
experience stress wardtype hospsize expcon zage
zgender zexperience zstress zwardtyi zhospsize
zexpcn cexpcon chospsize;
CLUSTER = hospital wardid;
WITHIN = age gender experience;
BETWEEN = (hospital) hospsize (wardid) expcon wardtype;
USEVARIABLES = stress expcon age gender experience
wardtype hospsize;
CENTERING = GRANDMEAN(expcon hospsize);
ANALYSIS: TYPE = THREELEVEL RANDOM;
ESTIMATOR = MLR;

```
MODEL:          %WITHIN%  
                stress ON age gender experience;  
                %BETWEEN wardid%  
                s | stress ON expcon;  
                stress ON wardtype;  
                %BETWEEN hospital%  
                s stress ON hospsize;  
                s; s WITH stress;  
OUTPUT:         TECH1 TECH8;  
SAVEDATA:       SAVE = FSCORES;  
                FILE = fs.dat;  
PLOT:           TYPE = PLOT2 PLOT3;
```

Model Results For Nurses Data

	Estimates	S.E.	Est./S.E.	Two-Tailed P-Value
WITHIN Level				
stress ON				
age	0.022	0.002	11.911	0.000
gender	-0.455	0.032	-14.413	0.000
experience	-0.062	0.004	-15.279	0.000
Residual Variances				
stress	0.217	0.011	20.096	0.000
BETWEEN wardid Level				
stress ON				
wardtype	0.053	0.076	0.695	0.487

Model Results For Nurses Data, Continued

	Estimates	S.E.	Est./S.E.	Two-Tailed P-Value
Residual Variances				
stress	0.109	0.033	3.298	0.001
BETWEEN hospital Level				
s ON				
hospsize	0.998	0.191	5.217	0.000
stress ON				
hospsize	-0.041	0.152	-0.270	0.787
s WITH				
stress	-0.036	0.058	-0.615	0.538

Model Results For Nurses Data, Continued

	Estimates	S.E.	Est./S.E.	Two-Tailed P-Value
Intercepts				
stress	5.753	0.102	56.171	0.000
s	-0.699	0.111	-6.295	0.000
Residual Variances				
stress	0.143	0.051	2.813	0.005
s	0.178	0.087	2.060	0.039

8.5 Cross-Classified Analysis

Students are cross-classified by school and neighbourhood at level 2.
An example with 33 students:

	School 1	School 2	School 3	School 4
Neighbourhood 1	XXXX	XX	X	X
Neighbourhood 2	X	XXXXX	XXX	XX
Neighbourhood 3	XX	XX	XXXX	XXXXXX

Source: Fielding & Goldstein (2006). Cross-classified and multiple membership structures in multilevel models: An introduction and review. Research Report RR 791, University of Birmingham.

- Regression analysis
- Path analysis (both subject and context are random modes)
 - Gonzalez, de Boeck, & Tuerlinckx (2008). A double-structure structural equation model for three-mode data. *Psychological Methods*, 13, 337-353
- SEM
- Longitudinal analysis (both subject and time are random modes)
- Random items (both subject and item are random modes)
- General idea: Two random modes
- Limited forms of multiple membership modeling (see Day 3 of Utrecht and the article Jeon & Rabe-Hesketh (2012). Profile-Likelihood Approach for Estimating Generalized Linear Mixed Models With Factor Structures. *JEBS*)

New Multiple Imputation Methods

- Multiple imputations for three-level and cross-classified data
- Continuous and categorical variables
- H0 imputations. Estimate a three-level or cross-classified model with the Bayes estimator. Not available as H1 imputation where the imputation model is setup as unrestricted model.
- The imputation model can be an unrestricted model or a restricted model. Restricted models will be easier to estimate especially when the number of clustering units is not large
- In the input file simply add the `DATA IMPUTATION` command

8.7 Applications to Item Response Theory Modeling

Random items, Generalizability Theory

- Items are random samples from a population of items
- The same or different items may be administered to individuals
- Suited for computer generated items and adaptive testing
- 2-parameter IRT model

$$P(Y_{ij} = 1) = \Phi(a_j\theta_i + b_j)$$

- $a_j \sim N(a, \sigma_a)$, $b_j \sim N(b, \sigma_b)$: random discrimination and difficulty parameters
- The ability parameter is $\theta_i \sim N(0, 1)$
- Cross-classified model. Nested within items and individuals. 1 or 0 observation in each cross-classified cell
- Interaction of two latent variables: a_j and θ_i
- The model has only 4 parameters - much more parsimonious than regular IRT models

9. Recent Mplus Papers

- Asparouhov & Muthén (2010). Computing the strictly positive Satorra-Bentler chi-square test in Mplus. Mplus Web Notes: No. 12
- Muthén (2010). Bayesian analysis in Mplus: A brief introduction. Technical Report. Version 3
- Muthén (2011). Applications of Causally Defined Direct and Indirect Effects in Mediation Analysis using SEM in Mplus.
- Muthén & Asparouhov (2011). LTA in Mplus: Transition probabilities influenced by covariates. Mplus Web Notes: No. 13
- Asparouhov & Muthén (2012). Using Mplus TECH11 and TECH14 to test the number of latent classes. Mplus Web Notes: No. 14
- Asparouhov & Muthén (2012). Comparison of computational methods for high dimensional item factor analysis
- Asparouhov & Muthén (2012). General random effect latent variable modeling: Random subjects, items, contexts, and parameters
- Asparouhov & Muthén (2012). Auxiliary variables in mixture modeling: A 3-step approach using Mplus. Mplus Web Note 15
- Asparouhov & Muthén (2012). Multiple group multilevel analysis. Web Note 16
- Muthén & Asparouhov (2012). Bayesian SEM: A more flexible representation of substantive theory. Psychological Methods, 17, 313-335
- Sobel & Muthén (2012). Compliance mixture modelling with a zero effect complier class and missing data. Biometrics, 68, 1037-1045
- Muthén & Asparouhov (2013). Item response modeling in Mplus: A multi-dimensional, multi-level, and multi-timepoint example
- Muthén & Asparouhov (2013). BSEM measurement invariance analysis. Web Note 17
- Asparouhov & Muthén (2013). Multiple-group rotational alignment. In preparation