Nonlinear Growth Models in M*plus* and SAS

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Nonlinear growth curves or growth curves that follow a specified nonlinear function in time enable researchers to model complex developmental patterns with parameters that are easily interpretable. In this article we describe how a variety of sigmoid curves can be fit using the M*plus* structural modeling program and the nonlinear mixed-effects modeling procedure NLMIXED in SAS. Using longitudinal achievement data, collected as part of a study examining the effects of preschool instruction on academic gain, we illustrate the procedures for fitting growth models of logistic, Gompertz, and Richards functions. Brief notes regarding the practical benefits, limitations, and choices faced in the fitting and estimation of such models are included.

Often a first task in studies of development is describing how individuals change (e.g., grow, decline, or both) over time (Wohlwill, 1973). Growth curve techniques and various extensions provide some of the necessary tools for modeling within-person changes and between-person differences in change (e.g., Bryk & Raudenbush, 1987, 1992; McArdle & Epstein, 1987; Rogosa & Willett, 1985; Singer & Willett, 2003). In recent years, as the breadth of substantive applications has widened, researchers have begun considering and seeking to use growth curve methods to describe complex patterns of nonlinear change (see McArdle & Nesselroade, 2003 for a review). In this article we present some background information on a set of growth curves that could be useful in describing longitudinal trends characterized by an elongated "S" or sigmoid shape, specifically, curves that follow the logistic, Gompertz, and Richards

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(generalized logistic) functions, and illustrate some procedures by which these models may be fit to empirical data.

As a companion to Ram and Grimm's (2007) introduction to nonlinear growth models, we further highlight how the M*plus* and SAS frameworks can be used to describe nonlinear changes with longitudinal time-structured data. We begin with a brief review of the basic growth curve modeling framework and an overview of how the framework may be used to describe patterns of nonlinear change. Subsequently, we introduce an example data set and illustrate how various sigmoid curves can be fit to the empirical data using both M*plus* and SAS, two popular statistical programs that complement each other because of their differing frameworks (i.e., structural equation and multilevel), and respective benefits and limitations (see Ghisletta & Lindenberger, 2004). We conclude with some notes regarding the practical benefits, limitations, and choices faced in the fitting and estimation of such models. Although a number of equations are included, the presentation is also meant to highlight the conceptual utility of the models and can be read as such.

GROWTH MODELING

Growth modeling (Browne, 1993; Cudeck, 1996; McArdle, 1986, 1988; Meredith & Tisak, 1990; Muthén & Curran, 1997; Rogosa & Willett, 1985) is a contemporary analytic technique for modeling systematic within-person change across a series of repeated measurements and between-person differences in those changes. Given repeated measurement of a variable, *Y,* for $n = 1$ to N participants on $t = 1$ to T occasions (or ages), a general form of the growth model can be written as

$$Y[t]_n = g_{0n} + g_{1n} \cdot A_1[t] + g_{2n} \cdot A_2[t] + \ldots + g_{kn} \cdot A_k[t] + e[t]_n, \tag{1}$$

where g_{0n} is the intercept for subject n (or predicted score when the vectors A_1 to A_k equal zero), g_{1n} to g_{kn} are individual slopes or the expected amount of change in Y for a one-unit change in A_1 to A_k, respectively, A_1 to A_k are vectors of basis coefficients indicating the relationship between the slopes and the observed scores, and $e[t]_n$ is a time-dependent residual that is uncorrelated with the intercept and slopes. The intercept and slopes are assumed to follow multivariate normal distributions with means, variances, and covariances

(e.g., $g_{0n}, g_{1n}, \ldots, g_{kn} \sim MVN \left(\begin{bmatrix} \mu_0 \\ \mu_1 \\ \vdots \\ \mu_k \end{bmatrix}, \begin{bmatrix} \sigma_0^2 \\ \sigma_{01} & \sigma_1^2 \\ \vdots & \vdots & \ddots \\ \sigma_{0k} & \sigma_{1k} & \cdots & \sigma_k^2 \end{bmatrix} \right)$). The time-dependent residuals are assumed to have a mean of zero, a single variance (homogeneity of residuals assumption), and to be unrelated to other variables and each other.

Conceptually, the basic growth modeling framework is used to capture the average trend or pattern of change over time and the between-person differences around the average trend. Practically, the framework fits within and bridges both structural equation modeling (SEM) and multilevel modeling traditions, and can be estimated in many SEM software packages, including M*plus* (Muthén & Muthén, 1998–2007), LISREL (Jöreskog & Sörbom, 1996), AMOS (Arbuckle & Wothke, 1999), Mx (Neale, Boker, Xie, & Maes, 1999), and EQS (Bentler, 1995)

and mixed-effects or multilevel programs, including HLM (Raudenbush, Bryk, Cheong, & Congdon, 2004), PROC MIXED in SAS (Littell, Milliken, Stroup, Wolfinger, & Schabenberber, 2006), SPSS MIXED, and LME in Splus.

In brief, within SEM, the basic growth model is fit as a restricted common factor model (Meredith & Tisak, 1990). The intercept, g_{0n}, and slopes, $g_{1n} - g_{kn}$, are latent factors indicated by the observed repeated measures. The factor loadings for the intercept are fixed at 1 and the loadings for the slope(s) (e.g., A_1 through A_k) define the shape or pattern of change. The intercept and slope factors have estimated means, variances, and covariances that, together with the residual variance, define the model's structural expectations for the observed covariance matrix and mean vector (see Grimm & McArdle, 2005).

Figure 1 is a path diagram of a growth model with an intercept and a slope. In the diagram, squares indicate manifest variables, circles indicate latent variables, and the triangle represents the unit constant. Directive relationships such as regression paths and factor loadings are represented as one-headed arrows; unanalyzed or symmetric relationships such as variances and covariances are represented as two-headed arrows; and unlabeled paths are fixed at 1. In this model there is a latent intercept, g_0, with unit factor loadings, and one latent slope, g_1, with loadings equal to the basis vector A_1. The intercept and slope have means (i.e., one-headed arrows from the constant), variances (i.e., two-headed arrows from and to the same variable), and a covariance (i.e., two-headed arrow connecting g_0 and g_1). The manifest variables have a single residual variance indicated by the common label (i.e., σ_e^2).

As a multilevel model, the basic growth model is fit as a two-level model (see e.g., Singer & Willett, 2003). At Level 1, the observed scores, $Y[t]_n$, are regressed on variables that define the functional form of within-person change, A_1 through A_k. At Level 2, fixed effects and random effects are captured by the means, variances, and covariances of the resulting person-

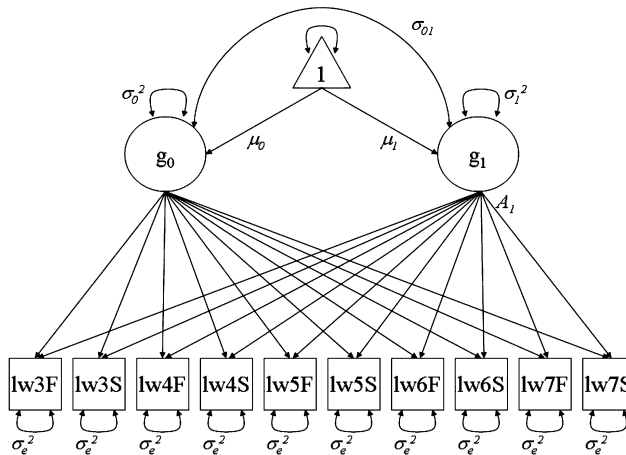FIGURE 1 Path diagram of a latent growth model. Squares indicate manifest variables, circles indicate latent variables, a triangle represents the unit constant, $\rightarrow$ indicate directive relationships, $\leftrightarrow$ indicate symmetric relationships (variances-covariances), g_0 is the latent intercept, g_1 is the latent slope, A_1 are the loadings for g_1, and unlabeled parameters are fixed at 1.

specific intercept, g_{0n}, and slope(s), g_{1n}–g_{kn}, obtained at Level 1. Readers can find further explication of the SEM–multilevel correspondence in recent literature (e.g., Chou, Bentler, & Pentz, 1998; Curran, 2003; MacCallum, Kim, Malarkey, & Kiecolt-Glaser, 1997; Willett, 2004). Additionally, Ferrer et al. (2004) described how basic growth models can be fit with a variety of multilevel and SEM programs and Ghisletta and Lindenberger (2004) provided a succinct discussion of the advantages and disadvantages of fitting growth models within each of these frameworks.

Key to the specification of the growth model, whether conceptualized as a structural equation or multilevel model, is the elements of the basis vectors, A_1 through A_k. These vectors (input as variables in the multilevel version of the model) are used to define a specific form of change. For example, if a linear pattern of change is desired, the elements of A_1 would be fixed to progress in a linear manner (e.g., 1, 2, 3, . . . ; "time" scores) and the elements of A_2 to A_k would be fixed to be zero. As presented shortly, more complex patterns of change are accommodated by fixing or adjusting the elements of the basis vectors to reflect the desired change pattern (e.g., Gompertz, logistic). Before getting to the specifics, however, a brief overview of the nonlinear models is presented.

NONLINEAR GROWTH MODELING

There are many ways in which the simple linear growth model can be expanded or adapted to describe nonlinear patterns of change over time (see also Ram & Grimm, 2007). One of the most common expansions is the addition of higher order polynomial terms to the linear growth model (e.g., Bryk & Raudenbush, 1992). For example, curvature in the change function might be accommodated by adding a quadratic term (e.g., $time^2$) or a cubic term (e.g., $time^3$) to the linear model. The linear model can be expanded by fixing the elements of A_2 to progress in a quadratic manner (e.g., 1, 4, 9, . . .), or the elements of A3 (e.g., 1, 8, 27, . . .). Interindividual differences in the higher order polynomial components are captured by the variance–covariance parameters associated with those latent variables (e.g., g_{2n}, g_{3n}, . . .). Other adaptations include the latent basis growth model (Meredith & Tisak, 1990), where the pattern or shape of nonlinear change is derived in an "exploratory," data-driven manner with minimal constraints on the elements of the "shape" vector (e.g., A_1). Such models are able to capture nonlinear forms of change over time. Conceptually, by either specifying or estimating the A_1 to A_k vectors, almost any "shape" of change, nonlinear or otherwise, can be accommodated in the growth modeling framework.

SIGMOID CURVES

Individual change may be characterized by accelerations and decelerations of a particular form. Learning and population growth, for instance, often consist of multiple "phases," an initial period of adjustment where little growth occurs, a rapid growth phase, and a slowdown as ability or population approaches task or environmental capacity limits (Thieme, 2003). Such patterns of growth can be described by sigmoid curves that generally look like an elongated S (see Figure 2). Key parameters of the mathematical functions used to describe such curves

include the lower and upper asymptotes, the rate of acceleration, the location of changes, and the symmetry (or asymmetry) in the pattern of acceleration and deceleration. Sigmoid curves have a long history of use in many areas of study, including biology, physiology, and economics (e.g., Westerfeld, 1956; Winsor, 1932), where they have been used to describe change processes ranging from bacterial growth to product innovation to early life increases in brain size. Within psychology, sigmoid functions have historically been used to model probability of binary outcomes (e.g., logistic regression), item response probabilities (e.g., item response characteristic curves), neuronal function (e.g., Easton, 2005), and learning (e.g., Browne & du Toit, 1991). Applications within the growth curve modeling framework, however, have been few (see however, Browne, 1993; McArdle, Ferrer-Caja, Hamagami, & Woodcock, 2002). Given the success of such functions for describing growth in many natural systems, we encourage further consideration and use of such functions to describe change in longitudinal panel data and investigate the intraindividual changes (and interindividual differences) therein. To foster such applications we illustrate how three types of sigmoid curves can be fit to longitudinal panel data using familiar growth curve modeling frameworks.

Before proceeding we draw attention to an important technical and practical distinction qualifying the "nonlinear" nature of nonlinear growth curves. Thus far we have used the term nonlinear to describe the pattern of observed changes with respect to time, not in reference to the characteristics of the parameters of the mathematical models describing these changes. Independent of the pattern of change over time, it must be noted that the mathematical functions describing intraindividual change come in (at least) two types, those that are linear in their parameters and those that are nonlinear in their parameters. In brief, the distinction has to do with the manner in which the interindividual differences or random effects are incorporated into the model. In looking at Equation 1 (or the path diagram in Figure 1), one can notice that the outcome variable, $Y[t]_n$, is a weighted sum of the interindividual difference variables (e.g., $g_{0n}, g_{1n}, g_{2n}, \ldots g_{kn}$; the random effects). The weighting of each variable is given by the vectors $A_1, A_2, \ldots, A_k$. When the elements of the A_k vectors are fixed parameters (i.e., invariant across persons) the random effects are *additive* and the model is *linear* in its parameters. For example, an exponential change pattern can be defined by setting the basis vector $A_1[t] = e^{-\alpha \cdot t}$. If the α parameter is the same for all participants, then the random effects of the model (i.e., g_{0n} and g_{1n}) are additive and the model is considered to be a model of nonlinear change that is linear in its parameters. In practice, however, the elements of the A_k vectors can take on almost any values, including those defined by a random, interindividual difference variable. Consider the case when $A_1[t] = e^{-\alpha_n \cdot t}$, where α_n is a variable that is allowed to differ between persons. In such cases, random effects are multiplied together (e.g., $g_{1n}(e^{-\alpha_n \cdot t})$), and are therefore *multiplicative,* which makes the model *nonlinear* in its parameters.

In this article we present models of both types, without drawing too much attention to the technical details and distinctions, other than indicating which software can be used for which type of model and some differences in substantive interpretation. Readers are referred to a set of excellent papers that cover the technical details of parameterization, estimation, and fitting of nonlinear growth curves that are either linear or nonlinear in their parameters (Blozis, 2004, 2007; Blozis & Cudeck, 1999; Browne, 1993; Browne & du Toit, 1991; Cudeck & du Toit, 2002; Pinheiro & Bates, 2000). We limit our focus to a didactic illustration of how various sigmoid growth curve models can be fit to empirical data using M*plus* (Muthén & Muthén, 1998–2007) and SAS PROC NLMIXED (Littell et al., 2006).

SIGMOID GROWTH FUNCTIONS

As noted earlier, nonlinear growth models can take many different forms (polynomials, latent basis, exponential, etc.). Here we focus on three S-shaped patterns of change: logistic, Gompertz, and Richards curves.

Logistic

The logistic function, as with all three curves covered here, is characterized by lower and upper asymptotes, and rates of change that are slowest near the asymptotes and fastest at an "inflection point" in the middle. The logistic growth model can be written as

$$Y[t]_n = g_{0n} + g_{1n} \cdot A_1[t] + e[t]_n$$

$$A_1[t] = \frac{1}{1 + e^{-(t-\lambda)\cdot\alpha}}. \tag{2}$$

where α denotes a rate of change and λ denotes the time at which the rate of change reaches its maximum, the inflection point. When α is positive, growth proceeds from g_{0n}, an individual-specific lower asymptote, to $g_{0n} + g_{1n}$, an individual-specific upper asymptote (and vice versa when α is negative). The defining feature of the logistic curve is that growth is distributed equally before and after the inflection point. That is, there is symmetry to the growth pattern such that exactly half of the total change has occurred before the inflection point. Figure 2a depicts logistic growth for three individuals with differences in their lower and long-term upper asymptotes (not always approached by $t = 10$). Despite these differences, however, all individuals are characterized by the same rate of change ($\alpha = .4$) and by the same inflection point ($\lambda = 5$), half of their total growth is achieved prior to the inflection point and half after. To highlight the symmetrical feature of the logistic curve, note that the individual depicted in bold, with lower asymptote at 0 and upper asymptote at 100, achieves 50% of her total growth at $t = 5$.

Gompertz

Similar in form to the logistic model, the Gompertz function is also characterized by upper and lower asymptotes and an inflection point. The Gompertz curve, however, is not symmetric with respect to its inflection point. Rather, growth proceeds in a manner such that roughly 37% (i.e., $1/e$) of the total growth occurs prior to the inflection point with the remainder occurring after. The model can be written as

$$Y[t]_n = g_{0n} + g_{1n} \cdot A_1[t] + e[t]_n$$

$$A_1[t] = e^{-e^{-\alpha(t-\lambda)}}, \tag{3}$$

where g_{0n} is the lower asymptote, $g_{0n} + g_{1n}$ equals the upper asymptotic value of the function, λ represents the time at which maximum growth rate occurs, and α is the rate of change. Figure 2b depicts Gompertz growth for three individuals who differ in their lower and upper asymptotes,
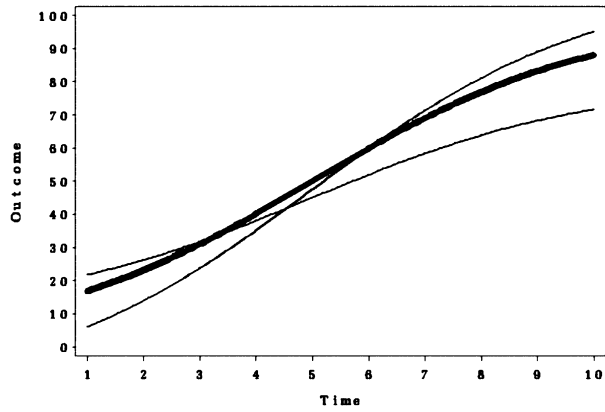
(a)
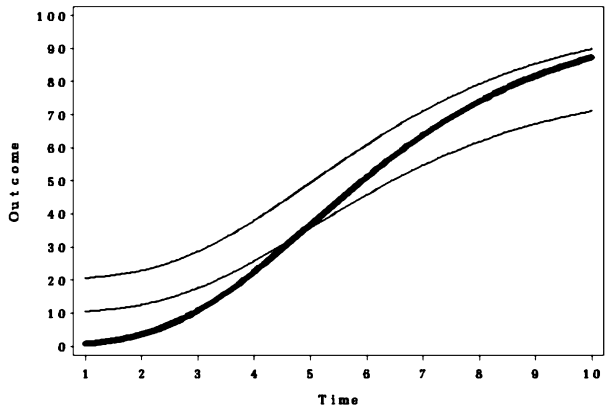
(b)
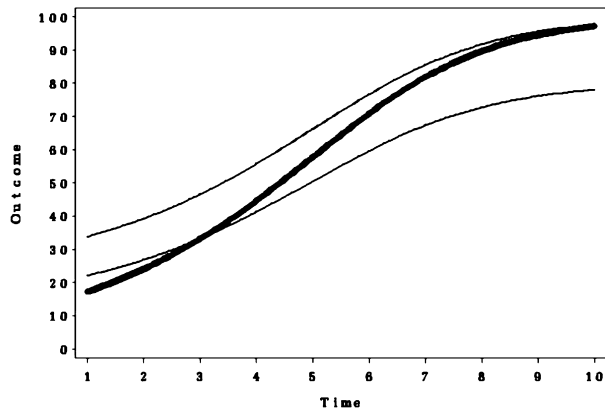
(c)

FIGURE 2   Example curves of (a) logistic growth model, (b) Gompertz growth model, and (c) Richards curve.

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but who are characterized by the same rate of change ($\alpha = .4$) and inflection point ($\lambda = 5$). The *asymmetrical* nature of Gompertz growth is highlighted in that the individual depicted in bold, with lower asymptote at 0 and upper asymptote at 100, achieves roughly 37% (i.e., $1/e$) of her total growth before the $\lambda = 5$ point in time, and the remaining 63% after. Substantively, it may be noted that the Gompertz growth curve, with its specific asymmetrical structure, is often used to describe the growth of populations in confined spaces with limited resources or nutrients (e.g., tumors; Laird, 1964). To the extent that the process one is interested in might follow the growth patterns found in "confined" biological or social systems (e.g., economic markets), the model might provide some of the sought-after links between behavioral and natural systems.

Richards

Both the logistic and the Gompertz curves have a priori defined symmetry or asymmetry around the inflection point. As a generalization of the logistic curve, the Richards curve (Richards, 1959) allows for flexibility in the asymmetry by including an additional parameter, τ, that controls which asymptote the point of inflection is nearest. This model can be written as

$$Y[t]_n = g_{0n} + g_{1n} \cdot A_1[t] + e[t]_n$$

$$A_1[t] = \frac{1}{(1 + \tau \cdot e^{-(t-\lambda)\cdot\alpha})^{\frac{1}{\tau}}}. \tag{4}$$

where g_{0n} is the lower asymptote, g_{1n} controls the upper asymptote, λ is the time at which α, the rate of change, is greatest, and τ controls whether this point of inflection is closer to the lower or upper asymptote. Together, these parameters allow for substantial flexibility in the shape of sigmoid curves. Figure 2c depicts Richards growth for three individuals who differ in their lower and upper asymptotes, but who are characterized by the same rate of change ($\alpha = .7$), inflection point ($\lambda = 5$), and relative asymmetry ($\tau = 2$). Although it is necessary for users to spend additional time understanding how differences in the parameters relate to different patterns of change, some general observations regarding how τ affects the shape of the curve are useful. As τ increases, the amount of change that occurs before the inflection point, λ, increases. More specifically, when $\tau < 1$, less than half the change occurs before λ (e.g., as in the Gompertz curve); when $\tau = 1$, half the change occurs before λ and half after (i.e., as in the logistic curve); when $\tau > 1$, more than half the change occurs before λ (e.g., as in Figure 2c). The latter asymmetrical possibility is highlighted in that the individual depicted in bold, with lower asymptote at 0 and upper asymptote at 100, achieves roughly 57% of her total growth before $t = 5$, and the remaining 43% after.

EXAMPLE DATA

To illustrate the use of these sigmoid growth functions we use data collected as part of a study examining the effects of preschool instruction on academic gains (Conner, Morrison, & Slominski, 2006). The data contain longitudinal test information on 383 children (195
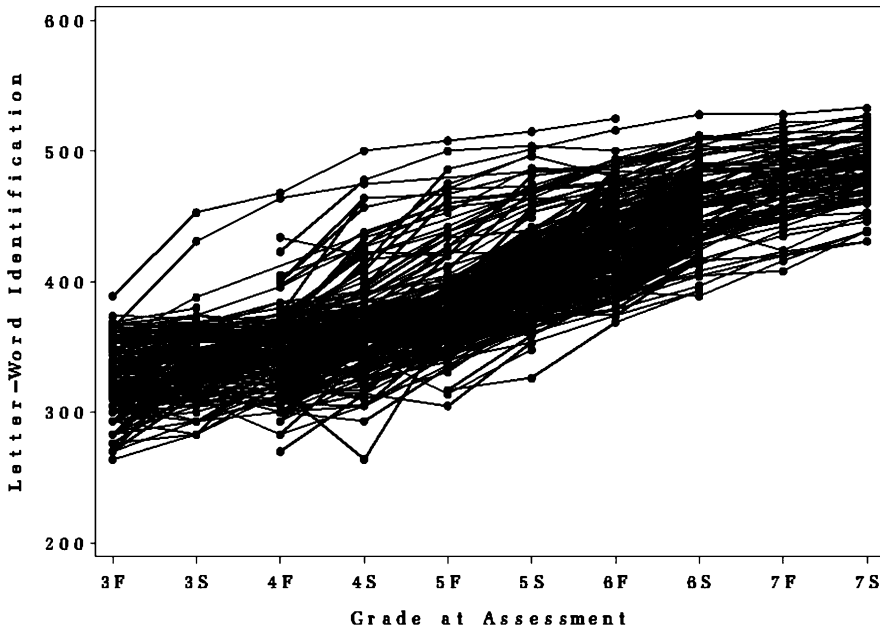
FIGURE 3   Longitudinal plot of individual trajectories on the Letter–Word Identification test of the Woodcock–Johnson Tests of Achievement.

girls, 188 boys) from an economically and ethnically diverse community, located on the urban fringe of a major Midwestern city in the United States. From the larger data set, we use the repeated assessments (10 occasions) of the Letter–Word Identification (LWID) subtest from the Woodcock–Johnson–III Test of Achievement (designed to measure letter and word recognition; McGrew, Werder, & Woodcock, 1991), that were collected in the fall and spring of each school year from preschool through second grade. Throughout the remainder of the article, these measures are referred to by the variable names lw3F–lw7S, denoting LWID W-scores scores starting with the fall score at age 3 (preschool) and ending with the spring score at age 7 (second grade). A longitudinal plot of the individual trajectories for these children is shown in Figure 3.

In examining the individual change patterns in Figure 3, the nonlinear change pattern is clear, and seems to be characterized by lower and upper asymptotes, and rates of change that are slowest near the asymptotes and fastest somewhere in the middle. In the following sections we describe how the logistic, Gompertz, and Richards growth models outlined earlier can be fit using M*plus* and SAS (PROC NLMIXED).

NONLINEAR GROWTH MODELS IN M*PLUS*

M*plus* (Muthén & Muthén, 1998–2007) is a general latent variable program that can be used to conduct a variety of statistical analyses including SEM, multilevel modeling, mixture modeling,

categorical data analysis, and combinations of such models. A typical M*plus* script contains six sections of commands: Title, Data, Variable, Analysis, Model, and Output. In this article we focus on the Model portion of the script where the statistical model is specified. We refer readers to the user's manual for information about the general layout and execution of the program (available from www.statmodel.com; Muthén & Muthén, 1998–2007). Appendix A also includes a full M*plus* script for a logistic growth model. For ease of reading, scripts are presented in the `Courier New` font to distinguish the program-specific commands from text, `CAPITAL` letters are used for M*plus* commands, and `lowercase` letters for manifest variables and latent variables that are specific to the data set and model.

In M*plus,* growth models can be specified in several ways—as a structural equation model with time-structured data, as a multilevel model using the the multilevel add-on, or using the `TSCORES` option when participants differ in the sampling of time. The `TSCORES` option is useful when participants vary in the sampling of time, as in an accelerated longitudinal study (see Bell, 1953; McArdle & Bell, 2000). With this option, the repeated observations are structured according to measurement occasion and timing variables that describe when the specific measurements took place (e.g., individuals' precise age at measurement) are also contained in the data. In this setup, the timing structure for the growth model is based on the timing scores (i.e., `TSCORES`) as opposed to measurement occasion. However, here we use the SEM component to specify the growth curves as restricted common factor models (see Figure 1). Note that the data file for such a specification must be in the wide format, with one record per person with a separate variable for each repeated measurement.

As a starting point for the presentation of nonlinear models we specify a linear growth model. The `MODEL` statement for the example data can be written as:

```
MODEL:
     g0 BY        lw3F-lw7S@1;
     g1 BY        lw3F@0       lw3S@1
                  lw4F@2       lw4S@3
                  lw5F@4       lw5S@5
                  lw6F@6       lw6S@7
                  lw7F@8       lw7S@9;
     g0 g1;       g0 WITH g1;       [g0* g1*];
     lw3F-lw7S (Ve);   [lw3F-lw7S@0];
```

The model is specified in three pieces. In the first piece, the elements of the basis vectors are specified. Specifically, two latent variables are specified, g0 and g1, that are indicated by the 10 observed scores (repeated measurement of the LWID) using the `BY` command. The loadings for the intercept, g0, are all fixed at 1 using the `@` symbol to denote a fixed parameter, and the loadings for the slope, g1, are fixed to follow a linear change pattern (i.e., `@0, @1, . . . , @9`). In a second part, the variances, covariances, and means of the latent factors are specified. Variances are denoted by listing the names of the factors (i.e., `g0 g1;`), a covariance is denoted using the `WITH` command (i.e., `g0 WITH g1;`), and means are denoted by listing the names of the factors within square brackets with asterisks to override a default that latent variables have fixed means equal to zero (i.e., `[g0* g1*];`). Finally, in the third part, the residual variances and intercepts of the observed variables are specified. Consistent with the homogeneity of variances assumptions, the residual variance is constrained to be equivalent

at all time points. This constraint is specified by listing the observed variable names with a common label, Ve, contained within parentheses (i.e., `lw3F-lw7S (Ve);`). The intercepts of the observed variables are fixed to 0 (i.e., `[lw3F-lw7S@0];`). This basic layout can then be expanded to accommodate nonlinear growth functions.

The extension from the linear model just given to nonlinear growth models, which are linear in their parameters, can be accomplished by introducing a series of constraints that "require" the slope loadings to follow a specific nonlinear function rather than the linear pattern given earlier. To accomplish this, two additional portions of script are needed: phantom variables and model constraints.

Phantom Variables

A *phantom variable* (also referred to as a node by Horn & McArdle, 1980) is a latent variable that is specified to be unrelated to every other variable in the model. Rather than being a formal part of the model, phantom variables can be used as "placeholders" for mathematical necessities (see Rindskopf, 1983). That is, the parameters associated with the phantom variable (i.e., mean, variance) can be specified to create dependencies between parameters in the model (i.e., constraints). In many instances, the parameters of phantom variables are used in conjunction with the `MODEL CONSTRAINT:` command to create nonequality mathematical constraints between parameters (see M*plus* user's manual, p. 28). In this context, phantom variables are used to create nonequality mathematical constraints between the slope loadings (elements of the basis coefficients) that "force" them to follow a specified nonlinear function.

To illustrate, a phantom variable can be created using the `BY` command. However, the relationship (e.g., factor loading) between the phantom variable and the observed variable is specified to be fixed at 0.[1] That is, to create the phantom variable, `phantom` (or any other name of choice), indicated by the first observed LWID variable, `lw3F`, we write

```
phantom BY lw3F@0;
```

Once created, `phantom`, like all other variables, has a set of parameters associated with it: a mean, a variance, and covariances with other variables. These parameters can then be used to "house" necessary model parameters. For example, the phantom variable can be used as a placeholder for the rate parameter in the logistic model, α, by attaching a label, '`alpha`,' to its mean.

```
[phantom*] (alpha);
```

---

[1]In version 4.0 and more recent versions of M*plus* the phantom variable could be specified without a manifest variable, such as
```
phantom BY ;
```
In version 5.0, the parameters of the nonlinear equations can simply be added in the `MODEL CONSTRAINT` command. Therefore creating phantom variables is unnecessary. This addition of parameters can be programmed as
```
MODEL CONSTRAINT:
NEW(alpha*.5 lambda*5);
```

Although the mean, variances, and covariances of phantom variables are all available as placeholders, we recommend using the phantom variable's mean, because this parameter can take on positive or negative values (whereas variances must be positive). The variance and covariances of the phantom variable are then "removed" by setting them to 0,

```
phantom@0;
phantom WITH g0@0 g1@0;
```

These four lines of script create a phantom variable that is unrelated to all other variables in the model, and has no variance, but has a mean labeled `alpha`. Additional phantom variables can be created and labeled as needed. For instance, for the logistic model, a second parameter, λ, can be introduced using the commands

```
phantom2 BY lw3F@0;
[phantom2*1] (lambda);
phantom2@0; phantom2 WITH g0@0 g1@0 phantom@0;
```

In this command, we specify a second phantom variable (`phantom2`), label its mean 'lambda,' and fix its variance and covariances to zero.

Model Constraints

The `alpha` and `lambda` parameters, created through the phantom variable procedure given previously, are then mathematically manipulated into their proper place in the model using constraints introduced with the `MODEL CONSTRAINT:` command. For the nonlinear growth models presented here, this means mathematically constraining the slope loadings (i.e., elements of the basis vectors, e.g., $A_k[t]$) to follow a prespecified nonlinear function (e.g., logistic, Gompertz, Richards).

To do this, the factor loadings associated with g1 are revised to be estimated and are labeled L1 to L10 (first step of the previous script);

```
g1 BY      lw3F* (L1)
           lw3S (L2)
           lw4F (L3)
           lw4S (L4)
           lw5F  (L5)
           lw5S (L6)
           lw6F (L7)
           lw6S (L8)
           lw7F (L9)
           lw7S (L10);
```

Note that an asterisk needs to be placed after the first variable to override the default that latent variables are indicated by the first variable with a fixed weight of 1. The `MODEL CONSTRAINT:` command is then used to specify the relationship between the slope loadings, now labeled L1 to L10, and the `alpha` and `lambda` parameters specified by the mathematical model. For example, to fit the logistic model of Equation 2 to the example data, where there are 10 equally spaced repeated observations, $t = 1$ to 10, the following constraints are necessary:

```
MODEL CONSTRAINT:
L1  = 1/(1 + EXP (-( 1-lambda)*alpha));
L2  = 1/(1 + EXP (-( 2-lambda)*alpha));
L3  = 1/(1 + EXP (-( 3-lambda)*alpha));
L4  = 1/(1 + EXP (-( 4-lambda)*alpha));
L5  = 1/(1 + EXP (-( 5-lambda)*alpha));
L6  = 1/(1 + EXP (-( 6-lambda)*alpha));
L7  = 1/(1 + EXP (-( 7-lambda)*alpha));
L8  = 1/(1 + EXP (-( 8-lambda)*alpha));
L9  = 1/(1 + EXP (-( 9-lambda)*alpha));
L10 = 1/(1 + EXP (-(10-lambda)*alpha));
```

Using these constraints, each factor loading (element of the basis vector) is specified to have the value that would be obtained by substituting the appropriate value of t into the logistic equation defining the pattern of change.

The phantom variable and model constraint capabilities can be used in conjunction to specify many nonlinear growth models (that are linear in their parameters). The M*plus* script for the logistic growth models is contained in Appendix A. Additionally, M*plus* and SAS scripts for all the models covered here are available online at http://psychology.ucdavis.edu/labs/Grimm/personal/downloads.html.

NONLINEAR GROWTH MODELS IN SAS

Nonlinear growth models can be fit in SAS using the NLMIXED procedure. PROC NLMIXED is a very flexible program that can be used to fit a wide variety of statistical models (e.g., item response models, see Sheu, Chen, Su, & Wang, 2005; survival models, see Lambert, Collett, Kimber, & Johnson, 2004; and shared parameter models, see Guo & Carlin, 2004) including many nonlinear growth models that are linear or nonlinear in their parameters. Here, we present programs that use the multilevel modeling framework (e.g., Level 1 & Level 2) and follow directly from the notation used previously.

First, we note that the data structure for growth modeling in NLMIXED is different from the structure used for M*plus* (Singer & Willett, 2003). Here, the data are in a long (i.e., relational, person-period) format with multiple records per person and variables for person identification, outcome measure, and time of assessment. For the example data, the identification variable is childid, the outcome measure is lw_w, and the time variable is time (ranging from 1–10).

We begin with a script for a linear growth model for illustration. This script is then expanded to articulate nonlinear growth models. As before, CAPITAL letters are used for SAS commands, and lowercase letters for manifest variables and latent variables that are specific to the data set and model. The statements beginning with '*' are comments that are helpful when programming complex models. An NLMIXED script for a linear growth model can be written as

```
PROC NLMIXED DATA = lw_long;
*specifying level-2 equations;
g_0n = mu_0 + d_0n;
g_1n = mu_1 + d_1n;
*specifying elements of the basis vector;
A1_t = time-1;
```

```
*specifying level-1 equation;
traject = g_0n + g_1n * A1_t;
*specifying model (outcome-its mean trajectory and residual variance);
MODEL lw_w ~ NORMAL(traject, v_e);
*specifying random effects;
RANDOM d_0n d_1n ~ NORMAL([0,0], [v_0, c_01, v_1])
SUBJECT = childid;
*specifying starting values;
PARMS
mu_0 = 300 mu_1 = 20
v_0 = 600 v_1 = 12 c_01 = 0
v_e = 175;
RUN;
```

The script begins by calling the NLMIXED procedure and `lw_long` data set. This is followed by two Level 2 equations of the growth model. The variable `g_0n`, the individual-level intercept, is set equal to the sample-level mean (`mu_0`) plus the individual deviation (`d_0n`) from the sample-level mean. The same type of Level 2 equation is written for the slope, `g_1n`. Next, the basis vectors or slope loadings, `A1_t` (i.e., $A_1[t]$), are defined. In the linear model they are set equal to `time-1`. This specifies that the elements of the basis vector proceed linearly with respect to the measurement occasions and centers the intercept at the first measurement occasion (see Ram & Grimm, 2007). This is followed by the Level 1 equation without the residual term. We call the expected true score of the outcome variable `traject` and, following the form of Equation 1, set it equal to the random intercept, `g_0n`, plus the random slope, `g_1n`, multiplied by the slope loadings, `A1_t`.

The next few lines of the script define the outcome variable, its distribution, and how the random effects should be included in the model. In the `MODEL` statement, the outcome variable (`lw_w`) is defined in terms of the Level 1 equation specified in the prior line and a residual term. Here, `lw_w` is specified to have a normal distribution with a mean equal to the expected value from the Level 1 equation, `traject`, and a Level 1 residual variance equal to `v_e`. Next, the random effects, or the Level 2 variances and covariances, are defined in the `RANDOM` statement. The individual deviations, `d_0n` and `d_1n`, for the intercept and slope (from the Level 2 equations) are specified to be multivariate normally distributed ($\sim$NORMAL) with means equal to 0 (`[0,0]`) and a variance-covariance matrix filled with parameters `v_0`, `c_01` and `v_1`, for the variance of `d_0n`, covariance between `d_0n` and `d_1n`, and variance of `d_1n`, respectively. Next, the identification variable (`childid`) is specified in the 'SUBJECT=' statement to indicate that random effects are across persons. Finally, the `PARMS` statement is used to set starting values for the estimation of all unknown parameters. Starting values can be difficult to generate, but are important for obtaining convergence within a reasonable time. Techniques that are useful for obtaining good starting values include (a) prefitting more simplistic versions of the model using PROC NLIN or NLMIXED without random effects, (b) prefitting models where the covariances among random effects are constrained to zero (by placing zeros in the covariance matrix on the `RANDOM` line), (c) reducing the number of quadrature points used in the estimation, and (d) using a first-order method of approximating the integral of the likelihood over the random effects (METHOD=FIRO) rather than maximum likelihood (METHOD=ML).

Adjusting the script for nonlinear growth models is straightforward. The equation for the basis vector is adjusted to match the desired model (e.g., logistic) and starting values for any additional parameters are added. For the logistic model, the specification of the basis vector can be programmed as

```
*specifying elements of the basis vector for a logistic model;
A1_t = 1/(1 + EXP(-(time-lambda)*alpha));
```

Starting values for the additional, fixed effect parameters are added in the PARMS statement,

```
alpha = .5 lambda = 5
```

As noted earlier, this logistic model only has additive random effects so it is linear in its parameters. One advantage of using NLMIXED is the opportunity to include parameters that enter nonlinearly into the model (multiplicative random effects). For instance, in the logistic model, we can specify the lambda parameter as random (varying across individuals) as opposed to fixed (invariant across individuals). The elements of the basis vector become $A_1[t] = \frac{1}{1+e^{-(t-\lambda_n)\cdot\alpha}}$, where λ has the subscript n to denote its value varies across individuals. In this model, lambda is a random effect and therefore has a mean, variance, and covariances with the other random effects. Inclusion of these additional random effects requires further adjustment of the NLMIXED script—specifically to the RANDOM line. Additionally, the following script is written with a single Level 1 equation, as opposed to the previous script that had separate Level 1 and Level 2 equations. This change is presented to show different ways to program growth models in NLMIXED—separate Level 1 and Level 2 equations could be specified and the model would be identical. Also, standard deviations and correlations are estimated as opposed to variances and covariances for ease of interpretation.

```
PROC NLMIXED DATA = lw_long;
traject = g_0n + g_1n * 1/(1 + EXP(-(time-lambda)*alpha));
MODEL lw_w ~ NORMAL(traject, s_e*s_e);
RANDOM g_0n g_1n lambda ~ NORMAL([mu_0, mu_1, mu_lambda],
[s_0*s_0,
s_0*r_01*s_1, s_1*s_1,
s_0*r_01ambda*s_lambda, s_1*r_11ambda *s_lambda, s_lambda*s_lambda])
SUBJECT = childid;
PARMS
mu_0 = 300 mu_1 = 200 mu_a = .05
s_0 = 15 s_1 = 25 s_lambda = 1
r_01 = 0 r_01ambda = 0 r_11ambda = 0
s_e = 13;
RUN;
```

The major change from the previous script is that the lambda parameter is now included on the RANDOM line with an associated mean (mu_lambda), standard deviation (s_lambda), and correlations with the intercept (r_01ambda) and the slope (r_11ambda). Currently, there are only a few programs, in addition to NLMIXED, that can be used to fit nonlinear growth models with multiplicative random effects (e.g., Mx, see Blozis, 2007; Splus using nlme, see Pinheiro & Bates, 2000; winBUGS, see Spiegelhalter, Thomas, Best, & Lunn, 2007). The SAS script for the logistic growth models is contained in Appendix B.

RESULTS

Nonlinear Growth Models

To illustrate how these models may be fit in practice, the series of nonlinear, sigmoid growth models (as well as a linear growth model) were fit to the example longitudinal achievement data using M*plus* and SAS. The parameter estimates and fit statistics from M*plus* and SAS are contained in Tables 1 and 2, respectively. Parameter estimates contained within the text reflect M*plus* estimates when possible. Predicted curves for each model are contained in Figure 4. Model fit was evaluated using common global fit indexes (e.g., comparative fit index [CFI],

TABLE 1
Parameter Estimates and Fit Statistics for the Linear and Nonlinear Growth Models
of Letter-Word Identification From M*plus*

	M1: Linear	*M2:* Logistic	*M3:* Gompertz	*M4:* Richards
Means (μ)				
$1 \rightarrow g_0$	308.64*	315.46*	323.03*	286.47*
$1 \rightarrow g_1$	19.49*	189.73*	224.18*	194.95*
Slope loadings ($A[t]$)				
$g_1 \rightarrow$ LW3F	=0	.06*	.01*	.19*
$g_1 \rightarrow$ LW3S	=1	.09*	.04*	.24*
$g_1 \rightarrow$ LW4F	=2	.15*	.09*	.30*
$g_1 \rightarrow$ LW4S	=3	.23*	.17*	.38*
$g_1 \rightarrow$ LW5F	=4	.34*	.26*	.48*
$g_1 \rightarrow$ LW5S	=5	.47*	.37*	.60*
$g_1 \rightarrow$ LW6F	=6	.61*	.47*	.74*
$g_1 \rightarrow$ LW6S	=7	.73*	.57*	.87*
$g_1 \rightarrow$ LW7F	=8	.82*	.66*	.96*
$g_1 \rightarrow$ LW7S	=9	.89*	.73*	.99*
Additional parameters				
α	—	.55*	.29*	1.38*
λ	—	6.21*	6.00*	6.87*
τ	—	—	—	5.92*
Variances and covariances (σ^2 & σ_{01})				
$g_0 \leftrightarrow g_0$	693.41*	597.44*	583.52*	680.62*
$g_1 \leftrightarrow g_1$	10.51*	719.92*	986.23*	787.51*
$g_0 \leftrightarrow g_1$	−37.49*	−193.98*	−195.82*	−322.55*
$e[t] \leftrightarrow e[t]$	212.20*	171.46*	178.78*	165.41*
Fit statistics				
χ^2/df	919/59	495/57	562/57	443/56
RMSEA	.195	.142	.152	.134
(90% C.I.)	(.184–.206)	(.130–.153)	(.141–.164)	(.123–.146)
CFI	.621	.807	.777	.829
TLI	.711	.848	.824	.863
−2LL	19,497	19,073	19,140	19,021
BIC	19,533	19,121	19,188	19,075
AIC	19,509	19,089	19,156	19,039

Note. = indicates the parameter was fixed at that value. χ^2 = maximum likelihood chi-square value; RMSEA = root mean squared error of approximation, CFI = comparative fit index; TLI = Tucker–Lewis Index, −2LL = −2 times the log likelihood value, BIC = Bayesian Information Criteria; AIC = Akaike Information Criteria.
* $p < .05$.

TABLE 2
Parameter Estimates and Fit Statistics for the Linear and Nonlinear Growth Models
of Letter-Word Identification From PROC NLMIXED in SAS

	M1: Linear	M2: Logistic	M3: Gompertz	M4: Richards	M5: Richards
Fixed effects					
$g_{0n}(\mu_{g0})$	308.64*	315.46*	323.03*	286.47*	292.09*
$g_{1n}(\mu_{g1})$	19.49*	189.73*	224.17*	194.95*	192.39*
$\alpha(\mu_\alpha)$	—	.55*	.29*	1.38*	1.46*
$\lambda(\mu_\lambda)$	—	6.20*	6.00*	6.87*	6.95*
$\tau(\mu_\tau)$	—	—	—	5.92*	5.59*
Random effects					
$g_0(\sigma^2_{g0})$	693.27*	597.31*	583.71*	680.69*	395.60*
$g_1(\sigma^2_{g1})$	10.50*	719.85*	985.96*	787.36*	506.15*
$\lambda(\sigma^2_\lambda)$	—	—	—	—	.89*
g_0/g_1 correlation $(\rho_{g0,g1})$	−.44*	−.30*	−.26	−.44*	−.74*
g_0/λ correlation $(\rho_{g0,\lambda})$	—	—	—	—	.22
g_1/λ correlation $(\sigma_{g1,\lambda})$	—	—	—	—	−.44*
$e(\sigma^2_e)$	212.28*	171.35*	178.76*	165.38*	143.52*
Fit statistics					
−2LL	19,497	19,073	19,140	19,021	18,820
Parameters	6	8	8	9	12
BIC	19,533	19,121	19,200	19,075	18,891
AIC	19,509	19,090	19,168	19,040	18,844

Note. Model 5 contained λ as a random parameter. Variances are the squares of the estimated standard deviations to assist in the comparison of parameter estimates. = indicates the parameter was fixed at specified value. −2LL = −2 times the log likelihood value; BIC = Bayesian Information Criteria; AIC = Akaike Information Criteria.
* $p < .05$.

Tucker–Lewis Index [TLI], and root mean squared error of approximation [RMSEA]) and model comparisons were made using likelihood-based indexes (e.g., Akaike's Information Criterion [AIC], Bayesian Information Certerion [BIC]). RMSEA values less than .05 were considered good, less than .08 were adequate, and less than .10 were marginal. Similarly, CFI and TLI values greater than .90 were considered adequate and values greater than .95 were considered good.

Linear (M1).   As a starting point, a linear growth model was fit to the data. The parameters of the linear model were $\mu_{g0} = 308.64$, $\mu_{g1} = 19.49$ indicating that, on average, children have a score of 308.64 in the fall of preschool and grow 19.49 units every half-year. Furthermore, the random effect parameters suggest significant interindividual differences in intercept, $\sigma^2_{g0} = 693.41$, and slope, $\sigma^2_{g1} = 10.51$, and that children who had lower intercepts tended to have greater rates of change from ages 3 through 7, $\sigma_{g0,g1} = -37.49$ ($\rho_{g0,g1} = -.44$). The predicted prototypical trajectory and expected individual trajectories for the linear curve are shown in Figure 4a. Overall, the linear growth model was found to be a relatively inadequate representation of the changes in letter and word recognition for this age range based on the global fit indexes: $\chi^2(9) = 919$, RMSEA = .195 (.184–.206), CFI = .621, TLI = .711, −2LL = 19,497, BIC = 19,533, AIC = 19,509). Thus, the parameter estimates and their representation of the data must be interpreted with caution.
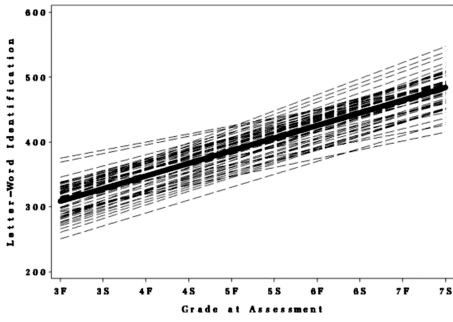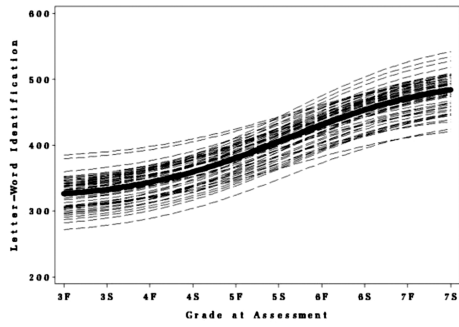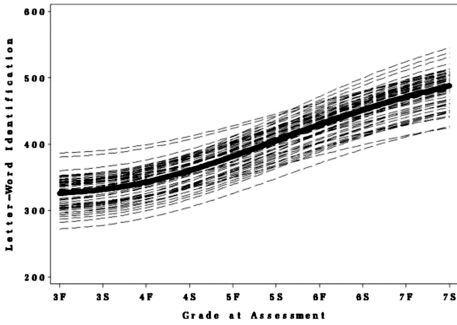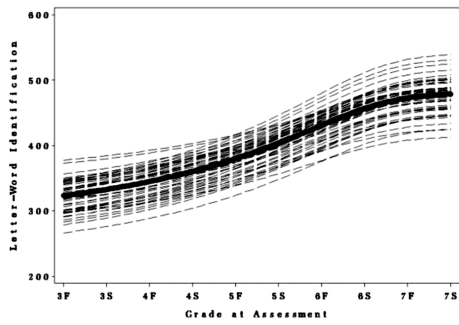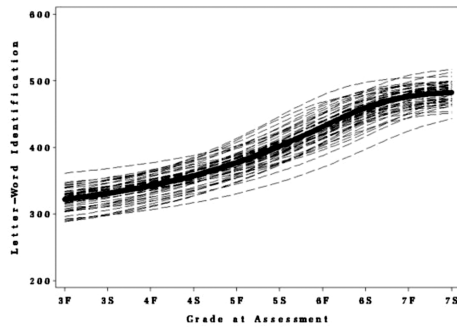
FIGURE 4 Mean and individual predicted growth trajectories based on the (a) linear, (b) logistic, (c) Gompertz, (d) Richards, and (e) Richards model with variation in λ.

Logistic (M2).   The logistic growth model, a sigmoid curve distinguished by its symmetry, was the first nonlinear curve fit to the example data. The parameters of the logistic curve were $\mu_0 = 315.46$, $\mu_1 = 189.73$, $\alpha = .55$, and $\lambda = 6.21$. Therefore, on average, the lower asymptote was 315.46; children grew 189.73 units to an upper asymptote of 505.19 (315.46 + 189.73); children reached half of their total change toward the end of kindergarten ($\lambda = 6.21$), and the growth rate was .55 ($\alpha = .55$). Furthermore, children varied in their lower asymptotic level ($\sigma_{g0}^2 = 597.44$), their predicted amount of change ($\sigma_{g1}^2 = 719.92$), and children who had a greater level of early reading achievement (lower asymptote) tended to show less total growth ($\sigma_{g0g1} = -193.98$; $\rho_{g0g1} = -.30$). The mean predicted trajectory and individual trajectories for the logistic curve are contained in Figure 4b. Note that the apparent discrepancy between the values for the upper asymptote in Table 1 and Figure 4B are due to the fact that the upper asymptote is the long-run upper bound that is not approached in this time interval. This discrepancy is common with sigmoid curves and can be seen as a limitation of these models as the expected amount of change is from the lower to the upper asymptotes, which might not be closely approached during the period of observation. The fit of the logistic model, $\chi^2(57) = 495$, RMSEA = .142 (.130–.153), CFI = .807, TLI = .848, $-2LL = 19,073$, BIC = 19,121, AIC = 19,089, was an improvement over the linear model based on the global fit indexes and likelihood statistics; however, the logistic model showed relatively poor fit based on the global fit indexes.

Gompertz (M3).   The next model fit to the data was the Gompertz, the defining feature of which is its a priori defined asymmetrical growth pattern. The parameters of the Gompertz model were $\mu_0 = 323.03$, $\mu_1 = 224.18$, $\alpha = .29$, and $\lambda = 6.00$ indicating that, on average, children have a lower asymptotic value of 323.03 and grew 224.18 units toward an asymptote of 547.21 (323.03 + 224.18). In the spring of kindergarten ($\lambda = 6.00$) students were changing more rapidly than at any other time and the growth rate was $\alpha = .29$. As with the logistic model, children varied in their lower asymptotic level ($\sigma_{g0}^2 = 583.52$), their predicted amount of change ($\sigma_{g1}^2 = 986.23$), and children who had a greater level of early reading achievement (lower asymptote) tended to show less growth ($\sigma_{g0g1} = -195.82$; $\rho_{g0g1} = -.26$). The mean predicted trajectory and individual trajectories for the Gompertz curve are contained in Figure 4c. As with the logistic model, small changes occurred as the children progressed through the first year of preschool, larger changes occurred in the second year of preschool, kindergarten, and into first grade before smaller changes were shown in second grade. The Gompertz model, $\chi^2(57) = 562$, RMSEA = .152 (.141–.164), CFI = .777, TLI = .824, $-2LL = 19,140$, BIC = 19,188, AIC = 19,156, fit better than the linear model, but not as well as the logistic model. This could be taken as an indication that the data are not characterized by assymetrical growth of the Gompertz type.

Richards curve (M4).   Moving toward a model where asymmetry of the growth pattern is estimated from the data, we next fit the Richards curve. The parameters obtained were $\mu_0 = 286.47$, $\mu_1 = 194.95$, $\alpha = 1.38$, $\lambda = 6.87$, $\tau = 5.92$. Therefore, on average, the lower asymptote was 286.47; children grew 194.95 units to the upper asymptote of 481.42 (286.47 + 194.95); children were changing most rapidly toward the beginning of first grade ($\lambda = 6.87$), the growth rate was 1.38 ($\alpha = 1.38$), and the growth was asymmetric, such that the majority of change occurred before the inflection point (i.e., $\tau = 5.92$). Furthermore, children varied

in their lower asymptotes ($\sigma_{g0}^2 = 680.62$), their predicted amount of change ($\sigma_{g1}^2 = 787.51$), and children who had a greater level of early reading achievement (lower asymptote) tended to show less growth ($\sigma_{g0g1} = -322.55$; $\rho_{g0g1} = -.44$). The mean predicted and individual trajectories for the Richards curve are contained in Figure 4d. The fit of the Richards curve, $\chi^2(56) = 443$, RMSEA $= .134$ ($.123–.146$), CFI $= .829$, TLI $= .863$, $-2LL = 19{,}021$, BIC $= 19{,}075$, AIC $= 19{,}039$, was an improvement over the linear, logistic, and Gompertz models based on the likelihood statistics and fit indexes. Therefore, the Richards curve was seen as the best representation (of the models fit) of the changes in letter and word recognition during this age period for these data. As with the previous models, the fit of the Richards curve remained relatively poor based on the global fit indexes, which suggests the changes in letter and word recognition are more complicated than these additive nonlinear models were able to capture. Multiplicative nonlinear models might be able to account for the additional heterogeneity in growth.

Richards curve with multiplicative random effects (M5).   A limitation in the preceding models is that the parameters of the nonlinear function (α, λ, and τ) are assumed to be invariant across persons (i.e., fixed). In NLMIXED, it is also possible to add further complexity into the model by allowing for interindividual differences in α, λ, and τ. For example, when allowing for interindividual differences in λ, the Richards growth curve becomes

$$Y[t]_n = g_{0n} + g_{1n} \cdot A_1[t] + e[t]_n$$

$$A_1[t] = \frac{1}{(1 + \tau \cdot e^{-(t-\lambda_n)\cdot\alpha})^{\frac{1}{\tau}}}. \tag{4a}$$

The expanded model accommodates another type of interindividual differences, but the added nonlinearity in parameters makes estimation computationally more difficult.

The Richards growth curve with interindividual differences in λ was fit in NLMIXED with a reduced number (i.e., 5) of quadrature points. The parameters obtained were $\mu_0 = 292.09$, $\mu_1 = 192.39$, $\alpha = 1.46$, $\mu_\lambda = 6.95$, and $\tau = 5.59$ (see Table 2). Similar to the prior model, the lower asymptote was 292.09; children grew 192.39 units to the upper asymptote of 484.48 ($292.09 + 192.39$); the growth rate was 1.46 ($\alpha = 1.46$), and the growth was asymmetric, such that the majority of change occurred before the inflection point (i.e., $\tau = 5.59$), and children were, on average, changing most rapidly toward the beginning of first grade ($\mu_\lambda = 6.95$). Now, the model includes interindividual differences in children's lower asymptotes ($\sigma_{g0}^2 = 395.60$), their predicted amount of change ($\sigma_{g1}^2 = 506.15$), and inflection point ($\sigma_\lambda^2 = .89$). These three random effects covary such that children who had a greater level of early reading achievement (lower asymptote) tended to show less growth ($\rho_{g0g1} = -.74$). The between-person differences in growth, in turn were negatively associated with the variation in λ_n, the inflection point ($\rho_{g1,\lambda} = -.44$) suggesting that children who begin to change earlier tended to change more. The mean predicted trajectory and individual trajectories for the expanded Richards curve are contained in Figure 4e. Although the mean trajectory did not visually change compared with the previous Richards growth curve, the additional interindividual differences can be seen in the individual curves.

The fit of the expanded Richards curve with variation in λ was obtained from NLMIXED ($-2LL = 18{,}820$, BIC $= 18{,}891$, AIC $= 18{,}844$) and was an improvement over all of the prior models. Using the likelihood of the data obtained from M*plus,* we were able to calculate the χ^2 statistic and RMSEA (Steiger & Lind, 1980) for this model using FITMOD.[2] The RMSEA for this multiplicative nonlinear Richards curve model was .096 (.082–.111), indicating a marginal fit and pointing to the importance of the variation in λ as a way to model these data. Further improvements in fit might be obtained by allowing for interindividual differences in α, and/or τ. However, given the amount of time spent obtaining model convergence (via good starting values) and the difficulties faced in the interpretation of multiplicative interindividual differences, we suggest careful consideration of practical and substantive issues before incorporating further complexity into these models.

DISCUSSION

Developmentalists are interested in describing how individuals change (e.g., grow, decline, or both) over time. As these descriptions increase in complexity, models of nonlinear change will be called on to provide more accurate, complete, and easily interpretable descriptions of how individuals change over time and interindividual differences in such change. In this article, we outlined how a selection of nonlinear sigmoid curves can be fit within a growth curve modeling framework to multiperson longitudinal data using M*plus* and SAS.

For the models fit in both programs, differences between the fixed-effect parameter estimates obtained from M*plus* and SAS were generally small; however, differences in the random-effect parameter estimates were noticeable for all nonlinear growth models fit in this project. Additional research is therefore necessary to assess when, and for what types of data, the random effect parameters in nonlinear growth models fit using these programs are accurate. Irrespective of the program used in the analyses conducted here, however, it should be noted that the substantive conclusions were identical.

At a more general level, we highlight some of the issues to be considered in choosing between M*plus* and SAS for the fitting of nonlinear growth curves (see also Chou et al., 1998; Curran, 2003; Ferrer et al., 2004; Ghisletta & Lindenberger, 2004; MacCallum et al., 1997; Willett, 2004). Advantages of using M*plus* are the advantages of using SEM. Notably, the nonlinear models can be combined with confirmatory factor models to accommodate measurement error (see Blozis, 2004; Hancock, Kuo, & Lawrence, 2001; McArdle, 1988); incomplete data can be handled on the outcome as well as in the predictors of change; change in the measurement instruments across time can be modeled (e.g., McArdle, Grimm, Hamagami, Bowles, & Meredith, 2009; McArdle & Hamagami, 2004); global fit statistics (e.g., CFI, RMSEA) are available to examine model fit and misfit; and multiple group and growth mixture models can be combined with nonlinear growth models to evaluate group (known or unknown) differences in longitudinal trajectories (Ram & Grimm, in press). Furthermore, estimation was quick and, in our case, was not very dependent on user-provided starting values; however, the

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[2]Available on request from Michael Browne, Department of Psychology, Ohio State University, Columbus, OH 43210-1222.

appropriate sign (positive or negative) of the nonlinear parameters was helpful. Additionally, extensions to multivariate growth models for examining correlated change are straightforward.

Major limitations of M*plus* are among the advantages of SAS PROC NLMIXED, including the possibility to fit nonlinear models with multiplicative random effects, which can substantially improve the fit compared to their additive counterparts (as was seen with these data). However, multiplicative models can be specified as additive models using a first-order Taylor series approximation (e.g., Browne & duToit, 1991), which can be fit using M*plus*, Lisrel, & Mx. Additionally, NLMIXED allows for flexibility in the timing basis. For example, age at assessment could be used as opposed to measurement occasion and each individual could have a unique (distinct) age at each assessment.[3] In turn, the limitations of NLMIXED are the advantages of M*plus* (global fit statistics, mixture modeling, quick estimation, etc.).

In conclusion, the nonlinear curves discussed here represent a sample of appropriate nonlinear models for longitudinal research. Other models of interest include the logarithmic, exponential, dual exponential (bi-exponential), and Michaelis–Menton curves, among many others. Ratkowsky's (1989) discussion of nonlinear regression models contains equations and descriptions for a variety of nonlinear regression models that can be adapted for growth curve analysis (see also Pinheiro & Bates, 2000, for applications in S and Splus). The flexibility of the growth modeling framework allows for a wide variety of nonlinear patterns of growth for examining within-person change. By outlining the ease with which such models can be implemented with currently available software we hope to have illustrated how useful the framework might be for describing the complexities of within-person change and between-person differences in change.

ACKNOWLEDGMENTS

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[3]Growth models with individually varying measurement occasions can be fit in M*plus* using the multilevel (TYPE = TWOLEVEL RANDOM) or TSCORES with the AT option. However, these options can only be used to fit polynomial (e.g., linear, quadratic, cubic) growth models. The multilevel option can fit growth models in a similar manner as linear multilevel modeling software (e.g., PROC MIXED) with data in the long format.

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APPENDIX A:
M*PLUS* SCRIPT FOR A LOGISTIC GROWTH MODEL

```
TITLE: Logistic Growth Model;
!Note: Comments begin with '!'
DATA:  FILE = wj_morrison.dat;
VARIABLE:
  NAMES =  childid lw3F lw3S lw4F lw4S lw5F lw5S lw6F lw6S lw7F lw7S;
  USEVARIABLES = lw3F - lw7S;
 MISSING = .;
ANALYSIS:
  TYPE = MEANSTRUCTURE MISSING H1;  ITERATIONS = 100000;  COVERAGE = 0;
MODEL:
g0     BY  lw3F - lw7S@1;
g1     BY  lw3F*1 (L1)
           lw3S (L2)
           lw4F (L3)
           lw4S (L4)
           lw5F (L5)
           lw5S (L6)
           lw6F (L7)
           lw6S (L8)
           lw7F (L9)
           lw7S (L10);
!Variances & Covariance
       lw3F - lw7S (Ve);
       g0*132 g1*20;  g0 WITH g1;
!Means
       [lw3F - lw7S @0];
       [g0*300 g1*153];
!Parameters of Logistic Model as a latent variables
       phantom1 BY lw3F@0;  phantom1@0;  [phantom1*3] (alpha);
       phantom1 WITH g0@0 g1@0; phantom2 BY lw3F@0;  phantom2@0;  [phantom2*1] (lambda);
       phantom2 WITH g0@0 g1@0 phantom1@0;
MODEL CONSTRAINT:
     L1    = 1/(1 + EXP (-( 1-alpha)*lambda));
     L2    = 1/(1 + EXP (-( 2-alpha)*lambda));
     L3    = 1/(1 + EXP (-( 3-alpha)*lambda));
     L4    = 1/(1 + EXP (-( 4-alpha)*lambda));
     L5    = 1/(1 + EXP (-( 5-alpha)*lambda));
     L6    = 1/(1 + EXP (-( 6-alpha)*lambda));
     L7    = 1/(1 + EXP (-( 7-alpha)*lambda));
     L8    = 1/(1 + EXP (-( 8-alpha)*lambda));
     L9    = 1/(1 + EXP (-( 9-alpha)*lambda));
     L10   = 1/(1 + EXP (-(10-alpha)*lambda));
OUTPUT:    SAMPSTAT STANDARDIZED;
```

APPENDIX B:
SAS SCRIPT FOR DATA RESTRUCTURING AND LOGISTIC GROWTH MODEL

```
*Reading data into SAS;
DATA morrison;
      INFILE 'D:\Nonlinear growth\Morrison\wj_morrison.dat' LINESIZE = 5000;
      INPUT childid age lw_w01 - lw_w10;
RUN;
*Restructuring Data from Wide to Long Format;
DATA morrison_long;
      SET morrison;
      lw_w = lw_w01;    time = 1;  OUTPUT;
      lw_w = lw_w02;    time = 2;  OUTPUT;
      lw_w = lw_w03;    time = 3;  OUTPUT;
      lw_w = lw_w04;    time = 4;  OUTPUT;
      lw_w = lw_w05;    time = 5;  OUTPUT;
      lw_w = lw_w06;    time = 6;  OUTPUT;
      lw_w = lw_w07;    time = 7;  OUTPUT;
      lw_w = lw_w08;    time = 8;  OUTPUT;
      lw_w = lw_w09;    time = 9;  OUTPUT;
      lw_w = lw_w10;    time = 10; OUTPUT;
      KEEP childid lw_w time;
RUN;
*Logistic Growth Model;
PROC NLMIXED DATA = morrison_long;
      g_0n = m_0 + d_0n;
      g_1n = m_1 + d_1n;
      A_t = 1/(1 + EXP(-(time - lambda)*alpha));
      traject = g_0n + g_1n * A_t;
      MODEL lw_w ~ NORMAL(traject, v_e);
      RANDOM d_0n d_1n ~ NORMAL([0,0], [v_0,
                                        c_01, v_1])
      SUBJECT = childid;
      PARMS
          m_0 = 300 m_1 = 200 v_0 = 620 v_1 = 900 c_01 = 0
          v_e = 175
          lambda = 5        alpha = .5;
RUN;
```