Standardized Residuals in Mplus

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1 Overview

The fit of structural equation models with normally distributed observed and latent variables can be evaluated by examining the normalized and standardized residuals computed in Mplus. These residuals are available for the ML, MLR, and MLF estimators and can be obtained by the *residual* output command. Suppose that Y is the vector of dependent observed variables and η is the vector of latent variables. The general structural equation model is given by

$$Y = \nu + \Lambda\eta + \varepsilon \tag{1}$$

$$\eta = \alpha + B\eta + \xi. \tag{2}$$

where ε and ξ are normally distributed residuals with 0 mean and covariance matrix Θ and Ψ respectively. The model implies a normal distribution for Y with mean

$$\mu = \nu + \Lambda(I - B)^{-1}\alpha \tag{3}$$

and covariance matrix

$$\Sigma = \Theta + \Lambda(I - B)^{-1}\Psi(I - B)^{-1\quad T}\Lambda^T \tag{4}$$

where I is the identity matrix. To evaluate the model fit we can compare the estimated $\hat{\mu}$ and $\hat{\Sigma}$ to the sample mean and m and covariance matrix S. If there is no missing data

$$m = \bar{Y} \tag{5}$$

and

$$S = \frac{1}{n}\sum_{i=1}^{n}(Y_i - \bar{Y})(Y_i - \bar{Y})^T \tag{6}$$

where n is the sample size. However if there is missing data m and S are estimated via the EM-algorithm.

The raw residuals computed in Mplus are

$$m - \hat{\mu} \tag{7}$$

$$S - \hat{\Sigma}. \tag{8}$$

The normalized residuals are computed as follows. The normalized mean residual for the $i-$th variable in the model is

$$\frac{m_i - \hat{\mu}_i}{\sqrt{Var(m_i)}} \tag{9}$$

where $Var(m_i)$ is the asymptotic variance estimate for the sample mean m_i. If there is no missing data

$$Var(m_i) = \frac{s_{ii}}{n}, \tag{10}$$

where s_{ii} is the sample variance for the $i-$th variable. If there is missing data $Var(m_i)$ is obtained from the inverse of the information matrix or the sandwich estimator depending on the model estimation method. The normalized covariance residual is given by

$$\frac{s_{ij} - \hat{\sigma}_{ij}}{\sqrt{Var(s_{ij})}}. \tag{11}$$

If there is no missing data

$$Var(s_{ij}) = \frac{s_{ii}s_{jj} + s_{ij}^2}{n} \tag{12}$$

see Bollen, page 259.

The standardized residuals are computed as follows. The standardized mean residual is

$$\frac{m_i - \hat{\mu}_i}{\sqrt{Var(m_i - \hat{\mu}_i)}}. \tag{13}$$

By Hausman's (1978) theorem, under the assumption of correct model specification

$$Var(m_i - \hat{\mu}_i) = Var(m_i) - Var(\hat{\mu}_i) \tag{14}$$

The standardized covariance residual is

$$\frac{s_{ij} - \hat{\sigma}_{ij}}{\sqrt{Var(s_{ij} - \hat{\sigma}_{ij})}} \tag{15}$$

and again by Hausman's (1978) theorem

$$Var(s_{ij} - \hat{\sigma}_{ij}) = Var(s_{ij}) - Var(\hat{\sigma}_{ij}) \tag{16}$$

We now briefly describe the computation of $Var(\hat{\sigma}_{ij})$ and $Var(\hat{\mu}_i)$. We use the delta method. Let θ denotes the parameter in the structural model. By the delta method

$$Var(\hat{\sigma}_{ij}) = \frac{\partial \sigma_{ij}}{\partial \theta} Var(\hat{\theta}) \left(\frac{\partial \sigma_{ij}}{\partial \theta} \right)^T \tag{17}$$

and similarly

$$Var(\hat{\mu}_i) = \frac{\partial \mu_i}{\partial \theta} Var(\hat{\theta}) \left(\frac{\partial \mu_i}{\partial \theta} \right)^T. \tag{18}$$

The $Var(\hat{\theta})$ matrix is obtained by the inverse of the information matrix or the sandwich estimator.

When covariates X are present in the structural model then $\hat{\mu}$ and $\hat{\Sigma}$ depend also on the mean and variance of the covariates μ_x and Σ_x as well as the parameters $\hat{\theta}$. In this case we augment the vector of parameter θ to include μ_x and Σ_x. The correlation between these two sets of parameters can be assumed to be 0 because the likelihood can be split as two separate parts

$$L(Y|X) + L(X). \tag{19}$$

The first part depends only on the θ parameters and the second only on μ_x and Σ_x. Thus the information matrix is block diagonal and we can assume 0 correlation between $\hat{\theta}$ and the X variable parameters. The asymptotic variance for μ_x and Σ_x is estimated using the maximum likelihood method, i.e., it is the inverse of the information matrix for the unrestricted mean and variance model.

2 Interpretation

The standardized and normalized residuals are essentially tests of model fit. If the model has p dependent variables essentially there are a total of

$p + p(p+1)/2$ tests formed by the normalized residuals and as many for the standardized. The tests are not independent. Proper interpretation of the results should include standard multiple testing theory. For example if there are more than 20 residuals, it is expected that at least one will have a value above 1.96, even if the model is correct. The residuals can also be used as a guide to model modifications, however, model modifications may be more effectively done with modification indices computed in Mplus.

Note also that the normalized residual is always smaller by absolute value than the standardized, i.e., the normalized residual is a more conservative test. Under the null hypothesis the standardized residual should have a standard normal distribution and any deviation from that would indicate model misfit. Under the null hypothesis the normalized residuals should have distribution smaller than the standard normal distribution and any deviation from that would indicate model misfit.

One problem with Hausman's (1978) approach to computing the residual variance is that sometimes the variance estimates given by (14) and (16) can be negative. In that case the standardized residual is not computed and Mplus prints 999. Typically in such situation the normalized residual can be used.

3 References

Bollen, K.A. (1989) Structural equations with latent variables. Wiley-Interscience.

Hausman, J. (1978), Specification tests in econometrics., Econometrica 46(6), 1251-71.