Quasi Maximum Likelihood Estimation of Structural Equation Models With Multiple Interaction and Quadratic Effects

Andreas G. Klein & Bengt O. Muthén[1]

This paper is currently under review in JASA (Journal of the American Statistical Association)

[1] Andreas G. Klein is Assistant Researcher at and Bengt O. Muthén is Professor at Graduate School of Education & Information Studies, Social Research Methodology, University of California Los Angeles, Los Angeles, CA 90095 (E-mail: *agklein@ucla.edu, bmuthen@ucla.edu)*. This research was supported under grant K02 AA 00230-01 from NIAAA and by NIMH and NIDA under grant No. MH40859.

Abstract

The development of statistically efficient and computationally practicable estimation methods for the analysis of structural equation models with multiple nonlinear effects has been called for by substantive researchers in psychology, marketing research, and sociology. But the development of efficient methods is complicated by the fact that a nonlinear model structure implies specifically nonnormal multivariate distributions for the indicator variables. In this paper, nonlinear structural equation models with quadratic forms are introduced and a new Quasi-Maximum Likelihood method for simultaneous estimation of model parameters is developed with the focus on statistical efficiency and computational practicability. The Quasi-ML method is based on an approximation of the nonnormal density function of the joint indicator vector by a product of a normal and a conditionally normal density. The results of Monte-Carlo studies for the new Quasi-ML method indicate that the parameter estimation is almost as efficient as ML estimation, whereas ML estimation is only computationally practical for elementary models. Also, the Quasi-ML method outperforms other currently available methods with respect to efficiency. It is demonstrated in a Monte-Carlo study that the Quasi-ML method permits computationally feasible and very efficient analysis of models with multiple latent nonlinear effects. Finally, the applicability of the Quasi-ML method is illustrated by an empirical example of an aging study in psychology.

Key words: structural equation modeling, quadratic form of normal variates, latent interaction effect, moderator effect, Quasi-ML estimation, variance function model.

1. INTRODUCTION

In the last two decades, structural equation modeling (SEM) has become a common statistical tool for modeling relationships between variables which cannot be observed directly, but only with measurement error. The relationships between these unobservable, latent variables are formulated in structural equations, and they are measured with errors by indicator variables in a measurement model. By the development of software packages for covariance structure analysis such as Amos (Arbuckle 1997), EQS (Bentler 1995; Bentler & Wu 1993), LISREL (Jöreskog & Sörbom 1993, 1996a), or Mplus (Muthén & Muthén 1998-2001), SEM has become available to a large community of researchers.

While ordinary SEM incorporates linear relationships among latent variables, models with nonlinear structural equations have recently attracted increasing attention. Several researchers have called for estimation methods for nonlinear latent variable models, and numerous substantive theories in education and psychology call for analysis of nonlinear models (Ajzen 1987; Ajzen & Fishbein 1980; Ajzen & Madden 1986; Cronbach 1975; Cronbach & Snow 1977; Fishbein & Ajzen 1975; Karasek 1979; Lusch & Brown 1996; Snyder & Tanke 1976). Also, a need for nonlinear extensions of ordinary SEM has been expressed from a methodological perspective (Aiken & West 1991; Busemeyer & Jones 1983; Cohen & Cohen 1975; Jaccard, Turisi & Wan 1990), and different ad-hoc estimation approaches have been developed. Hayduk (1987) established the estimation of an elementary interaction model with one latent product term proposed by Kenny and Judd (1984). Using LISREL 7 (Jöreskog & Sörbom 1989), they formed products of indicators for measuring the latent product term. Two-step LISREL approaches for this elementary model were proposed by Moosbrugger, Frank, and Schermelleh-Engel (1991) and Ping (1995, 1996a, 1996b, 1998) who implemented a stepwise LISREL procedure by estimating the measurement model in a first step and the parameters of the structural equation in a second step.

Other approaches aim at estimating the nonlinear model within the framework of covariance structure analysis. The technique of forming products of indicators was improved by Jaccard and Wan (1995), Jöreskog and Yang (1996, 1997), and Yang Jonsson (1997) who used nonlinear parameter constraints for estimation of the elementary interaction model under LISREL 8 (Jöreskog & Sörbom 1996a). Simulation studies show that the LISREL-ML estimation procedure can be used for parameter estimation of the elementary interaction model (Yang Jonsson 1997), but applicability seems to be limited to elementary quadratic or cross-product models because of instable sampling characteristics of the covariance matrices which include covariances of products of indicators. Also, the distributional assumptions of LISREL-ML are violated for a nonlinear structural equation model, and standard errors and χ^2-statistics can be erroneous and require an adjustment for bias (Yang-Wallentin & Jöreskog 2001). Moreover, simulation studies for the elementary interaction model indicate that the LISREL parameter estimators do not have maximum efficiency (Klein & Moosbrugger 2000; Schermelleh-Engel et al. 1998).

As an alternative to covariance structure analysis, a two-stage least squares (2SLS) estimation technique has been developed by Bollen (1995, 1996) and Bollen and Paxton (1998) using instrumental variables for estimating an elementary quadratic or interaction model, but, although no distributional assumptions are violated for this method, simulation studies showed that 2SLS estimators are substantially less efficient when compared to alternative estimation techniques (Klein & Moosbrugger 2000; Schermelleh-Engel et al. 1998). Using Bayesian estimation techniques, Arminger and Muthén (1998) proposed a computationally intensive method and demonstrated it for elementary models with one latent product term. Blom and Christoffersson (2000) developed an estimation method based on the empirical characteristic function of the distribution of the indicator variables. But both approaches seem to be limited to elementary models because of their computational burden.

Recently, Wall and Amemiya (2000) developed a two-step methods of moments (2SMM) technique for a general polynomial structural equation model. In the first step, the parameters of the measurement model are estimated by using a factor analytical technique. In the second step, conditional moments of products of latent variables under the condition of certain indicator variables are calculated and a methods of moments procedure using these conditional moments is applied to estimate the parameters of the structural equation. In particular, the assumption of normally distributed latent variables can be relaxed with this technique when distribution-free factor score estimators are used in the first step. Wall and Amemiya illustrate the 2SMM estimation technique with simulation studies for a quadratic and a cubic linear structural equation model with one latent independent variable measured with high reliability. In theory, the 2SMM approach can be generalized to more complex nonlinear latent variable models. But it needs to be investigated how much the non-simultaneous, two-step estimation procedure and the choice of a complex nonlinear model affect the efficiency of the 2SMM estimators in cases with low to medium reliable indicator variables.

With the LMS (latent moderated structural equations) method, Klein and Moosbrugger (2000) introduced a maximum likelihood estimation technique for latent interaction models with multiple latent product terms. In the LMS method, the latent independent variables and the error variables are assumed to be normally distributed. The distribution of the indicator variables is approximated by a finite mixture distribution and the loglikelihood function is maximized by use of the EM algorithm. In contrast to non-ML estimation methods, LMS also allows for likelihood ratio model difference tests which can test for the significance of one or several nonlinear effects simultaneously. Simulation studies for the elementary interaction model indicate that LMS provides efficient parameter estimators and a reliable model difference test, and they show no indication of bias of standard errors (Klein 2000; Klein & Moosbrugger 2000; Schermelleh-Engel et al. 1998). But, although models with multiple latent product terms can be analyzed with the LMS method, the method can become computationally too intensive for models with three or more product terms involved in the structural equation and many indicator variables involved in the measurement model.

In this paper, a Quasi-ML estimation method for structural equation models with quadratic forms is proposed. It has been developed for the efficient estimation of more complex nonlinear structural equation models, which cannot be analyzed with the LMS method because of the computational burden involved by a more complex nonlinear model structure. The development of an ML estimation procedure for a nonlinear structural equation model is confronted with the fact that the distribution of a polynomial of normal variates is nonnormal in general, which entails a nonnormal multivariate distribution for the indicator variables. For the Quasi-ML method proposed in this paper, an appropriate transformation is carried out which reduces the number of nonnormally distributed components of the original indicator vector to one nonnormally distributed component of the transformed indicator vector. After this transformation, the model is treated as a variance function model (Carroll, Ruppert & Stefanski 1995), and mean and variance functions for the nonlinear model are calculated. A quasi-likelihood estimation principle is applied and the nonnormal density function of the indicator vector is approximated by a product of an unconditionally normal and a conditionally normal density function. A Quasi-ML estimator is established by maximizing the loglikelihood function based on the approximating density function. Simulation studies show that the Quasi-ML estimates are very close to the Maximum Likelihood estimates given by the LMS method, and the Quasi-ML estimators are almost as efficient as the ML estimators. Also, the Quasi-ML estimator is substantially more efficient than the 2SMM estimator.

The contents of this paper are as follows. In section 2 (Structural Equation Models with Quadratic Forms), we introduce a general structural equation model with a quadratic form of latent variables. In section 3 (Quasi-ML Estimation Procedure), the transformation of the indicator vector, the calculation of mean and variance functions, and the Quasi-ML estimation procedure is developed. In section 4 (Simulation Studies), the finite sample properties of the Quasi-ML estimators with respect to bias and efficiency are examined, and the Quasi-ML model difference test is evaluated. In section 5 (Empirical Example), the applicability of the proposed Quasi-ML method to the analysis of empirical data sets is illustrated by an example.

2. STRUCTURAL EQUATION MODELS WITH QUADRATIC FORMS

In this section, a structural equation model with a general quadratic form of latent independent variables (predictor variables) is introduced. The elementary interaction model proposed by Kenny and Judd (1984) with one latent interaction effect and the model proposed by Klein and Moosbrugger (2000) with multiple interaction effects but no quadratic effects are special cases of the model specified here. The model we propose here covers structural equations with a polynomial of degree two of predictor variables, and it is a special case of the general polynomial structural equation model described by Wall and Amemiya (2000). We propose the following structural equation model with a quadratic form:

$$\eta_t = \alpha + \mathbf{\Gamma}\, \xi_t + \xi_t'\, \mathbf{\Omega}\, \xi_t + \zeta_t\,, \quad t = 1,...,\mathrm{N}, \tag{1}$$

where η_t is a latent dependent variable (criterion variable), α is an intercept term, ξ_t is a $(\mathrm{n} \times 1)$ vector of latent predictor variables, $\mathbf{\Gamma}$ is a $(1 \times \mathrm{n})$ coefficient matrix, $\mathbf{\Omega}$ is a symmetric $(\mathrm{n} \times \mathrm{n})$ coefficient matrix, and ζ_t is a disturbance variable. We call the parameters given by the matrices $\mathbf{\Gamma}$ and $\mathbf{\Omega}$ the structural parameters of the model. The quadratic form $\xi_t'\mathbf{\Omega}\xi_t$ of the structural equation (1) distinguishes the model from ordinary linear SEM. The vector ξ_t is assumed to be multivariate normally distributed with $E(\xi_t) = \mathbf{0}$ and $\mathrm{Cov}(\xi_t, \xi_t') = \mathbf{\Phi}$. The disturbance variable ζ_t

is assumed to be normally distributed with $E(\zeta_t) = \mathbf{0}$, $\text{Var}(\zeta_t) = \psi$, and $\text{Cov}(\zeta_t, \xi_t') = \mathbf{0}$. The latent variables of the structural equation are measured with error via measurement models:

$$\mathbf{x}_t = \mathbf{\Lambda}_x \xi_t + \delta_t \;, \quad t = 1,\ldots,N, \tag{2}$$

$$\mathbf{y}_t = \mathbf{\Lambda}_y \eta_t + \varepsilon_t \;, \quad t = 1,\ldots,N, \tag{3}$$

where $\mathbf{x}_t$ is a $(q \times 1)$ vector of observed indicators of ξ_t, $\mathbf{\Lambda_x}$ is a $(q \times n)$ factor loading matrix, and $\boldsymbol{\delta}_t$ is a $(q \times 1)$ vector of measurement errors. Similarly, $\mathbf{y}_t$ is a $(p \times 1)$ vector of observed indicators of η_t, $\mathbf{\Lambda_y}$ is a $(p \times 1)$ factor loading matrix, and $\boldsymbol{\varepsilon}_t$ is a $(p \times 1)$ vector of measurement errors. The error vectors $\boldsymbol{\delta}_t$ and $\boldsymbol{\varepsilon}_t$ are assumed to be multivariate normally distributed with $E(\boldsymbol{\delta}_t) = \mathbf{0}$, $E(\boldsymbol{\varepsilon}_t) = \mathbf{0}$, a diagonal covariance matrix $\text{Cov}(\boldsymbol{\delta}_t, \boldsymbol{\delta}_t') = \Theta_\delta$, a diagonal covariance matrix $\text{Cov}(\boldsymbol{\varepsilon}_t, \boldsymbol{\varepsilon}_t') = \Theta_\varepsilon$, $\text{Cov}(\boldsymbol{\delta}_t, \xi_t') = \mathbf{0}$, $\text{Cov}(\boldsymbol{\delta}_t, \boldsymbol{\varepsilon}_t') = \mathbf{0}$, and $\text{Cov}(\boldsymbol{\varepsilon}_t, \xi_t') = \mathbf{0}$, $\text{Cov}(\zeta_t, \boldsymbol{\delta}_t') = \mathbf{0}$, and $\text{Cov}(\zeta_t, \boldsymbol{\varepsilon}_t') = \mathbf{0}$. It is further assumed that the identifiability of the measurement model for the x-variables is guaranteed by certain restrictions on the parameters of the measurement model. For the identification of the measurement model for the y-variables, it is assumed that y_{1t} is a scaling indicator with loading $\lambda_{y11} = 1$. It is further assumed without loss of generality that the y-variables have zero mean, which implies that $\alpha = -\text{tr}(\mathbf{\Omega\Phi})$. This restriction does not limit the applicability of the model in practice. The nonlinear structural equation implies that the vector $\mathbf{y}_t$ is nonnormally distributed in general (Jöreskog & Yang 1996; Klein & Moosbrugger 2000).

3. QUASI-ML ESTIMATION PROCEDURE

The major characteristic of a nonlinear structural equation model lies in the fact that the latent dependent variable η_t given by the structural equation (Equation 1) is nonnormally distributed in general, which also implies that the vector $\mathbf{y}_t$ of indicator variables is nonnormally distributed. The distribution type of η_t has been investigated for special cases in the literature (Gurland 1955; Imhof 1961; Shah 1963; Press 1966), but, as Johnson and Kotz (1970) point out, these representations of the nonnormal density function are neither theoretically nor practically useful because of their complicated structure. The characteristic function of η_t or $\mathbf{y}_t$ can be expressed in closed form (Srivasta & Khatri 1979), but it is not possible to derive the density function from the characteristic function by integrating out the transform variable, because the integral cannot be solved in an analytically closed form.

The Quasi-ML estimation procedure developed in this section is based on an approximation of the nonnormal density function $f(x, y)$ of the indicator vector $(\mathbf{x}_t', \mathbf{y}_t')'$ by a nonnormal density $f^*(x, y)$, which is a product of a normal and a conditionally normal density. This approximation makes use of the concept of variance function models (Carroll et al. 1995), where the mean and variance function of a dependent variable conditional on the independent variables is specified. The maximization of the quasi-loglikelihood function derived from the approximating density $f^*(x, y)$ yields the Quasi-ML parameter estimates.

The organization of the remainder of this section is as follows. In subsection 3.1 (Transformation of Indicator Vector), a transformation of the nonnormally distributed indicator vector $(\mathbf{x}_t', \mathbf{y}_t')'$ is carried out such that only one component of the transformed indicator vector is nonnormally distributed. In subsection 3.2 (Calculation of Mean and Variance Function), the conditional mean and variance of the remaining nonnormally distributed component are derived. In subsection 3.3 (Quasi-ML Estimation), the derived mean and covariance function are used to develop the Quasi-ML estimation procedure. Also, the calculation of standard errors for the parameter estimates and the quasi-likelihood-ratio test statistic are explicated.

(IN THIS PREPRINT VERSION, WE DO NOT PROVIDE THE TECHNICAL DETAILS OF THE TRANSFORMATION (SECTION 3.1) AND THE CALCULATION OF MEAN AND VARIANCE FUNCTION (SECTION 3.2), ALTHOUGH IT APPEARS IN THE ORIGINALLY SUBMITTED PAPER. THIS IS BECAUSE OF INTENDED SOFTWARE IMPLEMENTATION OF THE ALGORITHM.)

3.1 Transformation of Indicator Vector

3.2 Calculation of Mean and Variance Function

3.3 Quasi-ML Estimation

In the Quasi-ML method proposed in this paper, the model parameters of the nonlinear structural equation model are simultaneously estimated by maximization of an approximating quasi-loglikelihood function. For the estimation of the model parameters, we apply the concept of quasi-likelihood estimation in variance function models (Carroll et al. 1995, pp. 269-272). This is based on the idea that the conditional distribution of $(y_1 \mid \mathbf{x} = x, \mathbf{u} = u)$ is approximated by a normal distribution. For this approximating normal distribution, the mean function $E[\mathrm{y}_{1t} \mid \mathbf{x}_t = x, \mathbf{u}_t = u]$ and the variance function $\mathrm{Var}(\mathrm{y}_{1t} \mid \mathbf{x}_t = x, \mathbf{u}_t = u)$ are used. The application of the quasi-likelihood principle suggests the following approximation $\mathrm{f}^*(x, y)$ of the density function $\mathrm{f}(x, y)$ (see Equation 4) of the indicator vector $(\mathbf{x}_t', \mathbf{y}_t')'$:

$$
\begin{aligned}
\mathrm{f}(x, y) &= \mathrm{f}_2(x, \mathbf{R}y)\, \mathrm{f}_3(y_1 \mid \mathbf{x}_t = x, \mathbf{u}_t = \mathbf{R}y) \\
&\approx \mathrm{f}_2(x, \mathbf{R}y)\, \mathrm{f}_3^*(y_1 \mid \mathbf{x}_t = x, \mathbf{u}_t = \mathbf{R}y) \\
&= \mathrm{f}^*(x, y)\,,
\end{aligned}
\tag{4}
$$

where $\mathrm{f}_2(x, u)$ is the normal density function of $(\mathbf{x}_t', \mathbf{u}_t')'$ and $\mathrm{f}_3^*(y_1 \mid \mathbf{x}_t = x, \mathbf{u}_t = \mathbf{R}y)$ is a univariate normal density with mean $E[\mathrm{y}_{1t} \mid \mathbf{x}_t = x, \mathbf{u}_t = \mathbf{R}y]$ and variance $\mathrm{Var}(\mathrm{y}_{1t} \mid \mathbf{x}_t = x, \mathbf{u}_t = \mathbf{R}y)$. It should be noted that the approximating density function $\mathrm{f}^*(x, y)$ is nonnormal in general. The Quasi-ML method maximizes the quasi-loglikelihood function based on the approximating density $\mathrm{f}^*(x, y)$ for the parameters vector $\boldsymbol{\theta}$ by application of standard numerical methods. Technically, this maximization is executed in two stages: In the first stage of the maximization process, the single-step iteration method (Isaacson & Keller 1966; Schwarz 1993) is used; in the second stage, the Newton-Raphson algorithm is applied. The maximization algorithm is programmed in Delphi Pascal program code and executed on an IBM compatible computer (Pentium III, 600 MHz).

For the computation of confidence intervals for the Quasi-ML parameter estimates, standard errors can be computed. The calculation of standard errors under Quasi-ML is straightforward and uses the ‘sandwich estimator’ $\mathbf{J}^*$ (Carroll et al. 1995), which estimates the covariance matrix of the Quasi-ML estimator. The sandwich estimator is given by

$$\mathbf{J}^* = \mathrm{N}^{-1}\ \mathbf{H}^{-1}\mathbf{J}\mathbf{H}^{-1}, \tag{5}$$

where N is the sample size, and **H** and **J** are the matrices

$$\mathbf{H} = \left(-E_{\hat{\theta}}\left[\frac{\partial^2 \ln f^*(\mathbf{x},\mathbf{y})}{\partial\theta_i \partial\theta_j}\right]\right), \quad \mathbf{J} = \left(E_{\hat{\theta}}\left[\left(\frac{\partial \ln f^*(\mathbf{x},\mathbf{y})}{\partial\theta_i}\right)\left(\frac{\partial \ln f^*(\mathbf{x},\mathbf{y})}{\partial\theta_j}\right)\right]\right). \tag{6}$$

The adjustment of the information matrix **J** given by Equation (14) is necessary in order to correct for a bias of estimation of standard errors. The entries of the matrices **H** and **J** are computed by stochastic integration, using a generated large sample of the indicator vector $(\mathbf{x}_t', \mathbf{u}_t')'$. Typically, a sample size between 20.000 and 50.000 is used, depending on the complexity of the model. This sample is generated according to the model equations specified by Equations (1, 2, 3), where the Quasi-ML estimates are chosen for the parameter values.

With the calculation of likelihood ratio test statistics based on the quasi-loglikelihood function, model difference tests can be carried out under the Quasi-ML method, and nested nonlinear structural equation models can be tested for significant differences in the model structure. For example, the statistical significance of multiple latent nonlinear effects can be tested simultaneously. In case of a nested model with r fixed parameters and parameter vector Θ_0, the log quasi-likelihood ratio $\mathrm{L} = -2(l(\hat{\Theta}_0) - l(\hat{\Theta}))$ converges in distribution to $\sum_{k=1}^{r} \lambda_k W_k$ when sample size increases (Carroll et al. 1995, pp. 265-267; Kent 1982). The $W_1,...,W_r$ are independently χ_1^2-distributed, and $\lambda_1,...,\lambda_r$ are calculated from the above matrices **H** and **J**. In the Quasi-ML method, the quantiles of the distribution of the test statistic are computed by simulation from the distribution of $\sum_{k=1}^{r} \lambda_k W_k$. Alternatively, algorithms given by Griffiths and Hill (1985) may be used.

4. SIMULATION STUDIES

The Quasi-ML method has been specifically developed for efficient and computationally inexpensive analysis of complex structural equation models with multiple nonlinear effects. In this section, the finite sample properties of the new Quasi-ML estimation method are examined by Monte-Carlo studies for two latent interaction models, the elementary interaction model proposed by Kenny and Judd (1984) and a complex interaction model with multiple latent nonlinear effects. In the first Monte-Carlo study, the new method is compared to three alternative techniques, the LMS method (Klein 2000; Klein & Moosbrugger 2000), the LISREL-ML method (Jöreskog & Yang 1996, 1997; Yang Jonsson 1997), and the 2SMM approach (Wall & Amemiya 2000). In the second Monte-Carlo study, a model with four latent nonlinear effects demonstrates the general applicability of the new Quasi-ML method to complex nonlinear latent variable models. The complex model is analyzed with the Quasi-ML approach only. The nonlinear part of this model is too complex to be analyzed with LMS or LISREL-ML. In principle, the complex model could also be analyzed by using the 2SMM method.

4.1 Monte-Carlo Study of Elementary Interaction Model

For the Monte-Carlo study, an elementary interaction model with the following structural equation is selected

$$\eta_t = \alpha + \gamma_1 \xi_{1t} + \gamma_2 \xi_{2t} + \omega_{12} \xi_{1t} \xi_{2t} + \zeta_t . \tag{7}$$

The following parameter values were selected: $\alpha = 1.00$, $\gamma_1 = 0.20$, $\gamma_2 = 0.40$, $\omega_{12} = 0.70$, $\phi_{11} = 0.49$, $\phi_{21} = 0.235$, $\phi_{22} = 0.64$, $\lambda_{x21} = 0.60$, $\lambda_{x42} = 0.70$, $\psi = 0.20$. The predictor variables ξ_{1t}, ξ_{2t} are measured by two x-variables each; ξ_{1t} is measured with reliabilities .49, .22, and ξ_{2t} is measured with reliabilities .64, .38. The criterion variable η_t is measured by one y-variable without error: $y_t = \eta_t$. The reliabilities were selected to be not very high in order to evaluate the performance of the methods under reasonably difficult conditions. The specified model has 14 parameters and 5 indicator variables. It has been investigated before by Jöreskog and Yang (1996), Klein and Moosbrugger (2000), and Schermelleh-Engel et al. (1998).

500 data sets of sample size N = 400 for the five indicator variables were generated with the

PRELIS program (Jöreskog & Sörbom 1996b). Each data set was analyzed with the new Quasi-ML approach, the LMS method, LISREL-ML, and 2SMM. In the first stage of 2SMM, we used the ML estimator for the measurement model parameters, as recommended by Wall and Amemiya (2000, p. 231) in case that the x-variables are normally distributed. The finite sample properties of the parameter estimators were examined by calculation of the mean and the standard deviation (MC-SD) of the estimates. The results are reported in Table 1 for the structural parameters γ_1, γ_2, and ω_{12}, which are the parameters of primary interest. In the study, there was no indication of bias of the parameter estimates for the structural parameters, so the means of the estimates are not reported here.

Table 1.

Estimation results of a Monte-Carlo study for an elementary interaction model (Equation 16). The columns for the structural parameters γ1, γ2, and ω12 show the true value, and the standard deviation (MC-SD) of parameter estimates for Quasi-ML, LMS, LISREL-ML, and 2SMM over 500 replications.

N = 400		Quasi-ML	LMS	LISREL-ML	2SMM
True Value		MC-SD	MC-SD	MC-SD	MC-SD
γ_1	0.200	0.074	0.064	0.089	0.095
γ_2	0.400	0.065	0.061	0.076	0.079
ω_{12}	0.700	0.108	0.094	0.161	0.139

For the evaluation of the new Quasi-ML method, it is of particular interest whether the maximization with respect to the loglikelihood function used in the Quasi-ML approach leads to less efficient estimators than the ML estimators provided by LMS. As was expected from the LMS method, Table 1 shows that it is the most efficient method. For the structural parameters, the standard deviations (MC-SD) under the Quasi-ML method are larger than under the LMS method (+15.6 % for γ_1, +6.6 % for γ_2, and +14.9 % for ω_{12}), but the differences are relatively small. The differences between LISREL-ML and LMS have been discussed in detail by Klein and Moosbrugger (2000) and Schermelleh-Engel et al. (1998). The standard deviations calculated in Table 1 for LISREL-ML and 2SMM are higher than they are for Quasi-ML and LMS. In particular, the interaction parameter ω_{12} is estimated less efficient by LISREL-ML and 2SMM: For LISREL-ML, the MC-SD of $\hat{\omega}_{12}$ is 71.3 % larger than for LMS, and for 2SMM the MC-SD of $\hat{\omega}_{12}$ is 47.9 % larger than for LMS. For $\hat{\omega}_{12}$, the results in Table 1 imply that the relative efficiency of 2SMM compared to Quasi-ML is 60 % and the relative efficiency of 2SMM is 46 % compared to LMS.

In the Monte-Carlo study, the estimation of standard errors was also investigated for the Quasi-ML method by calculating the 95 % coverage. The coverage of the true model parameter by the estimated 95 % confidence interval was 91.8 % for γ_1, 94.2 % for γ_2, and 94.4 % for ω_{12}. For evaluating the deviation of the coverage for γ_1 from the nominal 95 % it must be noted that the predictor ξ_1 is measured with a very low reliability only. Overall, the results indicate that the estimated Quasi-ML standard errors can be used for the construction of confidence intervals for the parameter estimates.

In the scope of the Monte-Carlo study carried out for the elementary model, the results lead to the conclusion that, given that the normality assumption for the x-variables is met, the Quasi-ML estimators are very close to the LMS estimators, and the Quasi-ML method can be expected to be almost as efficient as LMS. The results further indicate that LISREL-ML and 2SMM cannot be expected to be optimally efficient methods for the estimation of latent interaction effects in situations where the x-variables have only low reliability. Other Monte-Carlo studies carried out by the authors but not reported here show that the differences in efficiency between Quasi-ML, LMS, and 2SMM become smaller when the reliability of the x-variables is higher.

4.2 Monte-Carlo Study of Multiple Interaction Model

For a Monte-Carlo study of an interaction model with multiple nonlinear effects, the following structural equation model was selected:

$$\eta_t = \alpha + \begin{pmatrix} \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 \end{pmatrix} \begin{pmatrix} \xi_{1t} \\ \xi_{2t} \\ \xi_{3t} \\ \xi_{4t} \end{pmatrix} \tag{8}$$

$$+ \begin{pmatrix} \xi_{1t} & \xi_{2t} & \xi_{3t} & \xi_{4t} \end{pmatrix} \begin{pmatrix} 0 & \omega_{12}/2 & \omega_{13}/2 & 0 \\ \omega_{12}/2 & 0 & \omega_{23}/2 & 0 \\ \omega_{13}/2 & \omega_{23}/2 & 0 & 0 \\ 0 & 0 & 0 & \omega_{44} \end{pmatrix} \begin{pmatrix} \xi_{1t} \\ \xi_{2t} \\ \xi_{3t} \\ \xi_{4t} \end{pmatrix} + \zeta_t$$

This model has four latent predictor variables ξ_{1t}, ξ_{2t}, ξ_{3t}, ξ_{4t} and one latent criterion variable η_t. It includes four latent nonlinear effects among the predictor variables. The following parameter values were selected for the structural parameters: $\alpha = -0.18$, $\gamma_1 = 0.40$, $\gamma_2 = 0$, $\gamma_3 = 0.20$, $\gamma_4 = 0.50$, $\omega_{12} = 0.10$, $\omega_{13} = 0.30$, $\omega_{23} = -0.40$, $\omega_{44} = 0.30$, $\psi = 0.30$. The structural coefficients ω_{12}, ω_{13}, ω_{23}, ω_{44} were selected to have opposite signs, and it is of particular interest in this study how precisely the Quasi-ML method estimates and separates multiple nonlinear effects of opposite signs. The five latent variables were measured by two indicator variables each. Variable ξ_{1t} was measured with reliabilities .77 and .55, ξ_{2t} with reliabilities .77 and .62, ξ_{3t} with reliabilities .77 and .68, ξ_{4t} with reliabilities .77 and .73, and η_t was measured with reliabilities .86 and .79. The entries of the covariance matrix $\mathbf{\Phi}$ of the four predictor variables are given by $\phi_{11} = 1.00$, $\phi_{21} = 0.50$, $\phi_{22} = 1.00$, $\phi_{31} = 0.50$, $\phi_{32} = 0.80$, $\phi_{33} = 1.00$, $\phi_{41} = 0$, $\phi_{42} = 0$, $\phi_{43} = 0.20$, $\phi_{44} = 1.00$.

The specified model has 35 parameters and 10 indicator variables. 500 replications of data sets with sample size N = 500 were generated, and each of the 500 data sets was analyzed with the Quasi-ML approach. The sample size of N = 500 gives a ratio of only 14.3 observations per parameter for the selected model which can be regarded as a critically low ratio for a latent variable model with four simultaneous nonlinear effects. The estimation results for the structural parameters are given in Table 2.

Table 2.

Estimation results of a Monte-Carlo study for a complex interaction model (Equation 17). The columns give for the structural coefficients: the true value, the mean (M) of parameter estimates, the standard deviation (MC-SD) of parameter estimates, and the mean of the estimated standard errors (Est-SE) over 500 replications.

N = 500		Quasi-ML Method			
True Value		M	MC-SD	Est-SE	95 % Coverage
γ_1	0.400	0.396	0.056	0.052	93.0
γ_2	0.000	0.004	0.089	0.089	95.0
γ_3	0.200	0.202	0.094	0.093	94.8
γ_4	0.500	0.499	0.053	0.047	91.8
ω_{12}	0.100	0.103	0.088	0.090	95.6
ω_{13}	0.300	0.293	0.089	0.089	95.0
ω_{23}	-0.400	-0.395	0.051	0.049	94.2
ω_{44}	0.300	0.300	0.035	0.036	95.6

In the Monte-Carlo study, the distributions of the estimators of the structural coefficients showed no substantial deviation from normality. For the structural parameters of $\mathbf{\Gamma}$ and $\mathbf{\Omega}$ in particular, the skewness of the estimates ranged from –0.25 to 0.25, and the kurtosis ranged from –0.02 to 0.30. Also, the means (M) of the estimates in Table 2 show no substantial bias for the structural parameters.

For every data set, the Quasi-ML method also estimates the standard errors for the parameter estimates. The means of these estimated standard errors (Est-SE) were computed and compared to the standard deviation of the estimates (MC-DC). The results show no sign of substantial bias for the estimated standard errors, except a small underestimation for ω_{12} (-11.3%). But it should be noted that ω_{12} is the coefficient of the latent product term $\omega_{12}\xi_1\xi_2$ which models a very small interaction effect with $\mathrm{Var}(\omega_{12}\xi_1\xi_2)/\mathrm{Var}(\eta_t) = 0.01$. Additionally, the 95 % coverage was calculated (see Table 2). The reported values for the 95 % coverage are very close to the nominal value of 95 %. Thus, the results of Table 2 support the Quasi-ML method and confirm that multiple latent linear, interactive, and quadratic effects, unless they are of a very small size, can be individually detected and estimated with the Quasi-ML method for the nonlinear structural equation model (Equation 17) and the sample size N = 500 examined in this study.

Besides the estimation of standard errors, the Quasi-ML method has the capability to execute model difference tests, which allow for a simultaneous testing of latent effects. To evaluate the performance of the Quasi-ML model difference test, the combined statistical hypothesis that at least one of two parameters ω_{12} and ω_{13} is different from zero was examined. The performance of the Quasi-ML method was investigated for the combined hypothesis in an additional simulation study. In this study, the simulated model was specified as above, but with ω_{12} and ω_{13} set to zero. 500 data sets of sample size N = 500 were generated and analyzed with the Quasi-ML method. In the results of the Monte-Carlo study, the test statistic was significant in 3.8 % of the cases when computed for a nominal significance level of 5%. This result indicates that for the model complexity and sample size selected for this study, there is no inflation of Type I error of the model difference test, and the Quasi-ML method allows for a reliable testing of simultaneous latent interaction effects. Also, when the performance of the model difference is compared to the estimation of standard errors, the Monte-Carlo study suggests that in a fairly complex model as selected here, the model difference test seems to provide an evenly reliable instrument for statistical inference on structural coefficients than the standard errors.

5. EMPIRICAL EXAMPLE

This section covers an empirical example of an aging study in psychology for an interaction model with two latent product terms and seven indicator variables. The empirical data set was collected by Thiele (1998) who investigated age-related effects of coping strategies and the maintaining of well-being for middle-aged males. Specifically, Thiele examined the effect of subjectively perceived fitness (ξ_{1t}), objective fitness (ξ_{2t}), and flexibility in goal adjustment (ξ_{3t}) on the level of complaining about one's mental or physical situation (η_t). He formulated the interaction hypotheses that the linear effect of flexibility in goal adjustment on the complaint level is high when individuals have low values on the fitness scales, but that it is neutralized for individuals with high values on the fitness scales. For persons with a high level of subjective fitness, the flexibility of goal adjustment is supposed to have only a small or negligible effect on complaint level, whereas for persons with a low perceived availability of bodily resources, the flexibility of goal adjustment is expected to be an important factor for the level of complaining. The interaction hypotheses can be modeled by including appropriate product terms ($\xi_{1t}\xi_{3t}$ and $\xi_{2t}\xi_{3t}$) in a structural equation model. The following nonlinear structural equation model was selected for analysis:

$$\eta_t = \alpha + \begin{pmatrix} \gamma_1 & \gamma_2 & \gamma_3 \end{pmatrix} \begin{pmatrix} \xi_{1t} \\ \xi_{2t} \\ \xi_{3t} \end{pmatrix} + \begin{pmatrix} \xi_{1t} & \xi_{2t} & \xi_{3t} \end{pmatrix} \begin{pmatrix} 0 & 0 & \omega_{13}/2 \\ 0 & 0 & \omega_{23}/2 \\ \omega_{13}/2 & \omega_{23}/2 & 0 \end{pmatrix} \begin{pmatrix} \xi_{1t} \\ \xi_{2t} \\ \xi_{3t} \end{pmatrix} + \zeta_t . \quad (9)$$

The subjectively perceived fitness (ξ_{1t}) refers to the self-evaluation of the effectiveness with which one's body is functioning. It was measured by a split scale (x_1, x_2) of self-concept of bodily efficiency (Deusinger 1998). The objective fitness (ξ_{2t}) refers to the objective level of fitness. It was measured by lung volume (x_3). Flexibility in goal adjustment (ξ_{3t}) addresses the fact that individuals are more or less willing to adapt their goals to the limits given by their individual physical situation or health condition, which refers to the coping style of a person. The latent predictor variable ξ_{3t} was measured by splitting a flexibility scale from Brandstädter and Renner (1990) into two subscales (x_4, x_5). The complaint level (η_t) was measured by two indicators (y_1: psychological complaints, y_2: psychovegetative complaints) given by the complaint inventory of Degenhardt and Schmidt (1994). A data set of sample size N = 302 was examined for the seven indicator variables. The univariate skewness of the five x-variables was between -0.33 and 0.01, their univariate kurtosis was between –0.07 and 0.30. The indicator variables y_1 and y_2 were nonnormal with univariate skewness of 1.09 and 0.72, respectively; their univariate kurtosis was 0.89 and 0.43, respectively.

The data were z-standardized and analyzed with the Quasi-ML method. The parameter estimates for the measurement model gave the following reliabilities for the observed variables: .92 for x_1, .59 for x_2, 1.00 for x_3, .99 for x_4, .36 for x_5, .67 for y_1, and .74 for y_2. The estimated correlations between the three ξ-variables were between –0.08 and 0.25.

Table 3.

Parameter estimates, estimated standard errors, and standardized parameter estimates for the structural equation parameters provided by the Quasi-ML method.

Parameter	Parameter Estimate	Estimated Standard Error	Parameter Estimate for Completely Standardized Model
γ_1	-0.384	0.063	-0.451
γ_2	-0.101	0.045	-0.123
γ_3	-0.217	0.061	-0.264
ω_{13}	0.133	0.052	0.155
ω_{23}	0.038	0.045	0.047
ψ	0.402	0.060	0.604

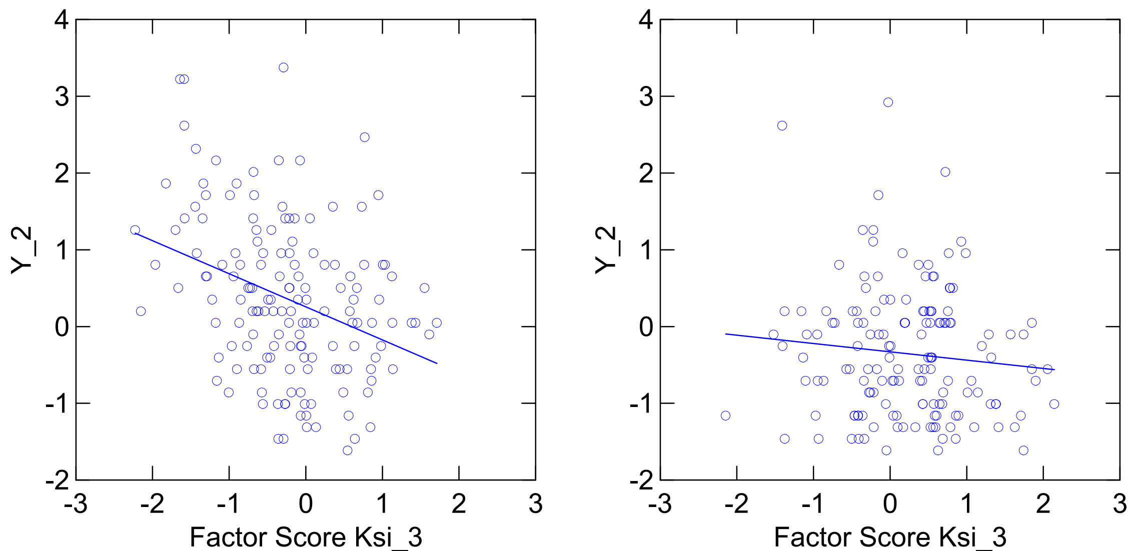
The negative signs of γ_1, γ_2, and γ_3 indicate the expected negative relationship between the latent variables ξ_{1t}, ξ_{2t}, ξ_{3t}, and η_t: High values for subjective or objective fitness and high values for flexibility in goal adjustment predict a low complaint level. The standard errors given in Table 3 show that the parameters γ_1, γ_2, and γ_3 are significantly different from zero at the 5% Type I error level. The positive signs of ω_{13} and ω_{23} indicate the expected direction of the interaction between fitness and flexibility in goal adjustment: Conditional on high levels of subjective (ξ_{1t}) or objective fitness (ξ_{2t}), the effect of flexibility in goal adjustment (ξ_{3t}) on complaint level (η_t) is low, that is, individual differences in complaint level cannot be predicted from individual differences in flexibility. Analogously, for low levels of subjective or objective fitness the flexibility level of goal adjustment has a substantial impact on complaint level.

A model difference test based on the likelihood-ratio test statistic confirms that the model with ω_{13} as a free parameter and ω_{23} fixed to zero fits the model significantly better than a linear model with both ω_{13} and ω_{23} fixed to zero. The chi-square value of this model difference test was $\chi^2_{diff,df=1} = 9.58$ with p-value less than 0.01. Setting further both ω_{13} and ω_{23} to be free parameters and comparing this model to the model where ω_{13} is free and ω_{23} is fixed to zero gives a chi-square value of $\chi^2_{diff,df=1} = 0.56$ for the likelihood-ratio test statistic, which is not significant at the 5% Type I error level.

The significant latent interaction effect between ξ_{1t} and ξ_{3t} can also be displayed graphically. For this purpose, factor scores for the ξ-variables were calculated by using the Quasi-ML parameter estimates for the measurement model of ξ_t. A median-split of the ξ_{1t} factor scores was carried out, and the sample was split into one half with low ξ_{1t} factor scores and one half with high ξ_{1t} factor scores. For each of the two subsamples, the scores of y_2 were plotted against the factor scores of ξ_{3t} (see Figure 1). The variable y_2 was selected for the scatterplots because it has a higher reliability than variable y_1. Figure 1 shows that the subjectively perceived fitness (ξ_{1t}) clearly moderates the relationship between flexibility in goal adjustment (ξ_{3t}) and complaint level (η_t): For the subsample with low factor scores for subjective fitness (left graph), the correlation is substantial ($corr(\xi_3, y_2) = -0.34$), whereas for the subsample with high factor scores for subjective fitness (right graph) the correlation is small ($corr(\xi_3, y_2) = -0.10$).

Figure 1.

Scatterplots for the factor scores of flexibility in goal adjustment (ξ_{3t}) and the observed scores for complaint level (y_2), displayed for individuals with low factor scores for subjective fitness (left graph) and high scores for subjective fitness (right graph). A regression line has been added to both scatterplots.



The analysis of the data set reveals that in the selected model subjectively perceived fitness both has a larger linear and interactive effect than objective fitness. Taking the high efficiency and the modeling capabilities of the Quasi-ML method into account, the new estimation method provides a reliable device for the analysis of structural equation models with multiple latent interaction effects.

6. SUMMARY

The Quasi-ML method proposed in this paper has been developed for an efficient and computationally feasible estimation of multiple nonlinear effects in structural equation models with quadratic forms. In the Quasi-ML approach, the nonnormal density function of the joint indicator vector is approximated by a product of a normal density and a conditionally normal density. It applies the concept of variance function models, and simulation studies for an elementary interaction model indicate that the Quasi-ML estimators are almost as efficient as ML estimators. Also, the simulation results suggest that the Quasi-ML method can be more efficient than the LISREL-ML method and the 2SMM method, particularly when a latent interaction model with low reliable indicator variables is analyzed. A second simulation study indicated that the Quasi-ML method is computationally feasible for a model with four simultaneous nonlinear effects and estimates interaction and quadratic effects very efficiently. Just as important, the Quasi-ML estimation of standard errors showed no substantial bias which supports precise significance testing of multiple nonlinear effects. Furthermore, the Quasi-ML method provides a test statistic for testing the significance of simultaneous effects in nested models. The applicability of Quasi-ML was demonstrated by an empirical example. Quasi-ML seems to be a very efficient, computationally feasible, and practically adequate approach, which is of particular relevance when rather complex structural equation models with several nonlinear effects are to be analyzed.

References

Aiken, L.S. & West, S.G. (1991). *Multiple regression: Testing and interpreting interactions.* Newbury Park: Sage Publications.

Ajzen, I. (1987). Attitudes, traits, and actions: Dipositional prediction of behavior in personality and social psychology. In L. Berkowitz (Ed.), *Advances in experimental social psychology* (Vol. 20, pp. 1-63). New York: Academic.

Ajzen, I. & Fishbein, M. (1980). *Understanding attitudes and predicting social behavior.* Englewood Cliffs, NJ: Prentice-Hall.

Ajzen, I. & Madden, T.J. (1986). Prediction of goal-directed behavior: Attitudes, intentions, and perceived behavioral control. *Journal of Experimental Social Psychology, 22,* 453-474.

Arbuckle, J.L. (1997). *AMOS users' guide version 3.6.* Chicago: Small Waters Corporation.

Arminger, G. & Muthén, B.O. (1998). A Bayesian approach to nonlinear latent variable models using the Gibbs sampler and the Metropolis-Hastings algorithm. *Psychometrika, 63*, 271-300.

Bentler, P.M. (1995). *EQS structural equations program manual*. Encino, CA: Multivariate Software.

Bentler, P. M. & Wu, E. J. C. (1993). *EQS/Windows user's guide.* Los Angeles: BMDP Statistical Software.

Blom, P. & Christoffersson, A. (2001). Estimation of nonlinear structural equation models using empirical characteristic functions. In R. Cudeck, S. Du Toit & D. Soerbom, *Structural Equation Modeling: Prsent and Future*. (pp. 443-460). Lincolnwood, IL: Scientific Software.

Bollen, K.A. (1995). Structural equation models that are non-linear in latent variables: A least squares estimator. In P.V. Marsden (Ed.), *Sociological methodology* (Vol. 25, pp. 223-251). Washington, DC: American Sociological Association.

Bollen, K.A. (1996). An alternative two stage least squares (2SLS) estimator for latent variable equations. *Psychometrika*, *61,* 109-121.

Bollen, K.A. & Paxton, P. (1998). Two-stage least squares estimation of interaction effects. In R.E. Schumacker & G.A. Marcoulides (Eds.), *Interaction and nonlinear effects in structural equation modeling.* (pp. 125-151). Mahwah, NJ: Erlbaum.

Brandstädter, J. & Renner, G. (1990). Tenacious goal pursuit and flexible goal adjustment: Explication and age-related analysis of assimilative and accomodative strategies of coping. *Psychology and Aging, 5*, 58-67.

Busemeyer, J.R. & Jones, L.E. (1983). Analyses of multiplicative combination rules when the causal variables are measured with error. *Psychological Bulletin, 93,* 549-562.

Carroll, R.J., Ruppert, D. & Stefanski, L.A. (1995). *Measurement error in nonlinear models* (1st ed). London: Chapman & Hall.

Cohen, J. & Cohen, P. (1975). *Applied multiple regression/correlational analysis for the behavioral sciences* (1st ed). Hillsdale, NJ: Erlbaum.

Cronbach, L.J. (1975). Beyond the two disciplines of scientific psychology. *American Psychologist, 30,* 116-127.

Cronbach, L.J. & Snow, R.E. (1977). *Aptitudes and instructional methods. A handbook for research on interactions.* New York: Irvington.

Degenhardt, A. & Schmidt, H. (1994). Physische Leistungsvariablen als Indikatoren für die Diagnose „Klimakterium Virile“ (Physical efficiency variables as indicators for the diagnosis of ‘climacterium virile’). *Sexuologie, 3,* 131-141.

Deusinger, I. (1998). *Frankfurter Körperkonzept-Skalen (Frankfurt bodily self-concept scales).* Göttingen: Hogrefe.

Fishbein, M. & Ajzen, I. (1975). *Belief, attitude, intention, and behavior: An introduction to theory and research.* Reading, MA: Addison-Wesley.

Griffiths, P. & Hill, I.D. (1985). *Applied statistics algorithms*. Horwood: London.

Gurland, J. (1955). Distribution of definite and of indefinite quadratic forms. *Annals of Mathematical Statistics, 26,* 122-127.

Hayduk, L.A. (1987). *Structural equation modeling with LISREL.* Baltimore: Johns Hopkins University Press.

Imhof, J.P. (1961). Computing the distribution of quadratic forms in normal variables. *Biometrika, 48,* 419-426.

Isaacson, E. & Keller, H.B. (1966). *Analysis of numerical methods*. New York: Wiley.

Jaccard, J., Turrisi, R. & Wan, C.K. (1990). *Interaction effects in multiple regression.* Newbury Park, CA: Sage publications.

Jaccard, J. & Wan, C.K. (1995). Measurement error in the analysis of interaction effects between continuous predictors using multiple regression: Multiple indicator and structural equation approaches. *Psychological Bulletin, 117*, 348-357.

Johnson, N.L. & Kotz, S. (1970). *Continuous univariate distributions –2.* New York: Wiley.

Jöreskog, K.G. & Sörbom, D. (1989). *LISREL 7: A guide to the program and applications* (2nd Ed.). Chicago, IL: SPSS.

Jöreskog, K.G. & Sörbom, D. (1993). *New features in LISREL 8*. Chicago, IL: Scientific Software.

Jöreskog, K.G. & Sörbom, D. (1996a). *LISREL 8: User's reference guide.* Chicago: Scientific Software.

Jöreskog, K.G. & Sörbom, D. (1996b). *PRELIS 2: User's guide.* Chicago: Scientific Software.

Jöreskog, K.G. & Yang, F. (1996). Non-linear structural equation models: The Kenny-Judd model with interaction effects. In G.A. Marcoulides & R.E. Schumacker (Eds.), *Advanced structural equation modeling* (pp. 57-87). Mahwah, NJ: Erlbaum.

Jöreskog, K.G. & Yang, F. (1997). Estimation of interaction models using the augmented moment matrix: Comparison of asymptotic standard errors. In W. Bandilla & F. Faulbaum (Eds.), *SoftStat '97* (Advances in Statistical Software 6, pp. 467-478). Stuttgart: Lucius & Lucius.

Karasek, R.A. (1979). Job demands, job decision latitude, and mental strain: Implications for job redesign. *Administrative Quarterly, 24*, 285-307.

Kenny, D.A. & Judd, C.M. (1984). Estimating the nonlinear and interactive effects of latent variables. *Psychological Bulletin, 96,* 201-210.

Kent, J.T. (1982). Robust properties of likelihood ratio tests. *Biometrika*, *69*, 19-27.

Klein, A. (2000). *Moderatormodelle. Verfahren zur Analyse von Moderatoreffekten in Strukturgleichungsmodellen (Moderator Models. Methods for the analysis of moderator effects in structural equation models)*. Hamburg: Dr. Kovac.

Klein, A. & Moosbrugger, H. (2000). Maximum likelihood estimation of latent interaction effects with the LMS method. *Psychometrika, 65 (4),* 457-474.

Krickeberg, K. & Ziezold, H. (1988). *Stochastische Methoden (Stochastic Methods)*. Berlin: Springer.

Longford, N.T. (1995). Random coefficient models. In G. Arminger, C.C. Clogg & M.E. Sobel (Eds.), *Handbook of Statistical Modeling for the Social and Behavioral Sciences* (pp. 519-577). New York: Plenum.

Lusch, R.F. & Brown, J.R. (1996). Interdependency, contracting, and relational behavior in marketing channels. *Journal of Marketing, 60*, 19-38.

Moosbrugger, H., Frank, D. & Schermelleh-Engel, K. (1991). Zur Überprüfung von latenten Moderatoreffekten mit linearen Strukturgleichungsmodellen (Estimating latent interaction effects in structural equation models). *Zeitschrift für Differentielle und Diagnostische Psychologie, 12,* 245-255.

Muthén, B.O. & Muthén, L. (1998-2001). *MPlus user's guide*. Los Angeles: Muthén & Muthén.

Ping, R.A. (1995). A parsimonious estimating technique for interaction and quadratic latent variables. *Journal of Marketing Research, 32,* August, 336-347.

Ping, R.A. (1996a). Latent variable interaction and quadratic effect estimation: A two-step technique using structural equation analysis. *Psychological Bulletin, 119*, 166-175.

Ping, R.A. (1996b). Latent variable regression: A technique for estimating interaction and quadratic coefficients. *Multivariate Behavioral Research, 31,* 95-120.

Ping, R.A. (1998). EQS and LISREL examples using survey data. In R.E. Schumacker & G.A. Marcoulides (Eds.), *Interaction and nonlinear effects in structural equation modeling.* (pp. 63-100). Mahwah, NJ: Erlbaum.

Press, S.J. (1966). Linear combinations of non-central chi-square variates. *Annals of Mathematical Statistics, 37,* 480-487.

Schermelleh-Engel, K., Klein, A. & Moosbrugger, H. (1998). Estimating nonlinear effects using a Latent Moderated Structural Equations Approach. In R.E. Schumacker & G.A. Marcoulides (Eds.), *Interaction and nonlinear effects in structural equation modeling.* (pp. 203-238). Mahwah, NJ: Erlbaum.

Schwarz, H.R. (1993). *Numerische Mathematik (Numerical Mathematics).* Stuttgart: Teubner.

Shah, B.K. (1963). Distribution of definite and of indefinite quadratic forms from a non-central normal distribution. *Annals of Mathematical Statistics, 34*, 186-190.

Snyder, M. & Tanke, E.D. (1976). Behavior and attitude: Some people are more consistent than others. *Journal of Personality, 44,* 501-517.

Srivasta, M.S. & Khatri, C.G. (1979). *An introduction to multivariate statistics*. New York.

Thiele, A. (1998). *Verlust körperlicher Leistungsfähigkeit: Bewältigung des Alterns bei Männern im mittleren Lebensalter (Loss of bodily efficacy: The coping of aging for men of medium age).* Idstein: Schulz-Kirchner-Verlag.

Wall, M.M. & Amemiya, Y. (2000). Estimation for polynomial structural equation models. *Journal of the American Statistical Association*, *95*, 929-940.

Yang Jonsson, F. (1997). *Non-linear structural equation models: Simulation studies of the Kenny-Judd model.* Uppsala, Sweden: University of Uppsala.

Yang-Wallentin, F. & Joereskog, K.G. (2001). Robust standard errors and chi-squares for interaction models. In G.A. Marcoulides & R.E. Schumacker & (Eds.), *New Developments and Techniques in Structural Equation Modeling.* (pp. 159-171). Mahwah, NJ: Erlbaum.