Non-Normal Growth Mixture Modeling

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Overview

1. A new GMM method
   - Examples of skew distributions
   - Normal mixtures
   - Introducing mixtures of non-normal distributions
   - Cluster analysis with non-normal mixtures
   - Non-normal mixtures of latent variable models:
     - GMM of BMI in the NLSY multiple-cohort study
     - GMM of BMI in the Framingham data
     - Math and high school dropout in the LSAY study
     - Cat’s cradle concern
   - Disadvantages and advantages of non-normal mixtures
   - Mplus specifications

2. A new SEM method: Non-normal SEM
   - Path analysis
   - Factor analysis
   - SEM

References:
Asparouhov & Muthén (2014). Non-normal mixture modeling and SEM. Mplus Web Note No. 19. - More to come
Examples of Skewed Distributions

- Body Mass Index (BMI) in obesity studies (long right tail)
- Mini Mental State Examination (MMSE) cognitive test in Alzheimer’s studies (long left tail)
- PSA scores in prostate cancer studies (long right tail)
- Ham-D score in antidepressant studies (long right tail)
Body Mass Index (BMI): \( \frac{kg}{m^2} \)

Normal 18 < \( BMI < 25 \), Overweight 25 < \( BMI < 30 \), Obese > 30
### NLSY Multiple-Cohort Data Ages 12 to 23

Accelerated longitudinal design - NLSY97

<table>
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<tr>
<th>Year</th>
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<th>14</th>
<th>15</th>
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**Totals**: 1,165 1,819 3,547 5,255 6,680 7,272 8,004 7,759 6,280 4,620 2,997 1,389

NLSY, National Longitudinal Survey of Youth

**Source**: Nonnemaker et al. (2009). Youth BMI trajectories: Evidence from the NLSY97, *Obesity*
BMI at Age 15 in the NLSY (Males, \( n = 3194 \))

Descriptive statistics for BMI15_2:

- \( n = 3194 \)
- Mean: 23.104
- Min: 13.394
- Variance: 20.068
- 20%-tile: 19.732
- Std dev.: 4.480
- 40%-tile: 21.142
- Skewness: 1.475
- Median: 22.045
- Kurtosis: 3.068
- 60%-tile: 22.955
- % with Min: 0.03%
- 80%-tile: 25.840
- % with Max: 0.03%
- Max: 49.868
Mixtures for Male BMI at Age 15 in the NLSY

- Skewness = 1.5, kurtosis = 3.1
- Mixtures of normals with 1-4 classes have BIC = 18, 658, 17, 697, 17, 638, 17, 637 (tiny class)
- 3-class mixture shown below

![Graph showing BMI distribution with 3 classes]
Several Classes or One Non-Normal Distribution?

- Pearson (1895)
- Hypertension debate:
  - Platt (1963): Hypertension is a "disease" (separate class)
  - Pickering (1968): Hypertension is merely the upper tail of a skewed distribution
- Schork et al (1990): Two-component mixture versus lognormal
- Bauer & Curran (2003): Growth mixture modeling classes may merely reflect a non-normal distribution so that classes have no substantive meaning
- Muthén (2003) comment on BC: Substantive checking of classes related to antecedents, concurrent events, consequences (distal outcomes), and usefulness
- Multivariate case more informative than univariate
What If We Could Instead Fit The Data With a Skewed Distribution?

- Then a mixture would not be necessitated by a non-normal distribution, but a single class may be sufficient.
- A mixture of non-normal distributions is possible.
Introducing Mixtures of Non-Normal Distributions in Mplus Version 7.2

In addition to a mixture of normal distributions, it is now possible to use

- Skew-normal: Adding a skew parameter to each variable
- T: Adding a degree of freedom parameter (thicker or thinner tails)
- Skew-T: Adding skew and df parameters (stronger skew possible than skew-normal)

References

- Arellano-Valle & Genton (2010): extended skew-t
- McNicholas, Murray, 2013, 2014: skew-t as a special case of the generalized hyperbolic distribution
Skew T-Distribution Formulas

Y can be seen as the sum of a mean, a part that produces skewness, and a part that adds a symmetric distribution:

\[ Y = \mu + \delta |U_0| + U_1, \]

where \( U_0 \) has a univariate t and \( U_1 \) a multivariate t distribution. Expectation, variance (\( \delta \) is a skew vector, \( \nu \) the df):

\[
E(Y) = \mu + \delta \frac{\Gamma\left(\frac{\nu-1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)} \sqrt{\frac{\nu}{\pi}},
\]

\[
Var(Y) = \frac{\nu}{\nu - 2} (\Sigma + \delta \delta^T) - \left( \frac{\Gamma\left(\frac{\nu-1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)} \right)^2 \frac{\nu}{\pi} \delta \delta^T
\]

Marginal and conditional distributions:

- Marginal is also a skew-t distribution
- Conditional is an extended skew-t distribution
Examples of Skew-T Distributions

Percentiles for I in Class 2:
- 5%-tile: 18.026
- 10%-tile: 18.656
- 25%-tile: 19.916
- 50%-tile: 21.717
- 75%-tile: 24.327
- 90%-tile: 27.658
- 95%-tile: 30.448

Percentiles for I in Class 1:
- 5%-tile: 22.169
- 10%-tile: 22.586
- 25%-tile: 23.835
- 50%-tile: 26.333
- 75%-tile: 30.220
- 90%-tile: 35.494
- 95%-tile: 39.936
Skewness = 1.5, kurtosis = 3.1
- Mixtures of normals with 1-4 classes: BIC = 18,658, 17,697, 17,638, 17,637 (tiny class). 3-class model uses 8 parameters
- 1-class Skew-T distribution: BIC = 17,623 (2-class BIC = 17,638). 1-class model uses 4 parameters
AIS data often used in the statistics literature to illustrate quality of cluster analysis using mixtures, treating gender as unknown.

Murray, Brown & McNicholas forthcoming in Computational Statistics & Data Analysis: "Mixtures of skew-t factor analyzers": How well can we identify cluster (latent class) membership based on BMI and BFAT?

Compared to women, men have somewhat higher BMI and somewhat lower BFAT.

Non-normal mixture models with unrestricted means, variances, covariance.
Table: Comparing classes with unknown gender

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<thead>
<tr>
<th>Normal 2c:</th>
<th>Normal 3c:</th>
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<tbody>
<tr>
<td>LL = -1098, # par.'s = 11, BIC = 2254</td>
<td>LL = -1072, # par.'s = 17, BIC = 2234</td>
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<tr>
<td><strong>Class 1</strong></td>
<td><strong>Class 2</strong></td>
</tr>
<tr>
<td>Male</td>
<td>78</td>
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<tr>
<td>Female</td>
<td>0</td>
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<table>
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<tr>
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<th>T 2c:</th>
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<tbody>
<tr>
<td>LL = -1069, # par.'s = 15, BIC = 2218</td>
<td>LL = -1090, # par.'s = 13, BIC = 2250</td>
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<tr>
<td><strong>Class 1</strong></td>
<td><strong>Class 2</strong></td>
</tr>
<tr>
<td>Male</td>
<td>84</td>
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<tr>
<td>Female</td>
<td>1</td>
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<thead>
<tr>
<th>Skew-T 2c:</th>
<th>PMSTFA 2c:</th>
</tr>
</thead>
<tbody>
<tr>
<td>LL = -1068, # par.'s = 17, BIC = 2227</td>
<td>BIC = 2224</td>
</tr>
<tr>
<td><strong>Class 1</strong></td>
<td><strong>Class 2</strong></td>
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<tr>
<td>Male</td>
<td>95</td>
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<td>Female</td>
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<th>PMSTFA 2c:</th>
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<tr>
<td><strong>Class 1</strong></td>
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<tr>
<td>Male</td>
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<tr>
<td>Female</td>
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</table>
Reduces the number of $\mu_c$, $\Sigma_c$ parameters for $c = 1, 2, \ldots C$ by applying the $\Sigma_c$ structure of an EFA with orthogonal factors:

$$
\Sigma_c = \Lambda_c \Lambda'_c + \Theta_c
$$

This leads to 8 variations by letting $\Lambda_c$ and $\Theta_c$ be invariant or not across classes and letting $\Theta_c$ have equality across variables or not (McNicholas & Murphy, 2008).

Interest in clustering as opposed to the factors, e.g. for genetic applications.
Non-Normal Mixtures of Latent Variable Models

Models:

- Mixtures of Exploratory Factor Models (McLachlan, Lee, Lin; McNicholas, Murray)
- Mixtures of Confirmatory Factor Models; FMM (Mplus)
- Mixtures of SEM (Mplus)
- Mixtures of Growth Models; GMM (Mplus)

Choices:

- Intercepts, slopes (loadings), and residual variances invariant?
- Scalar invariance (intercepts, loadings) allows factor means to vary across classes instead of intercepts (not typically used in mixtures of EFA, but needed for GMM)
- Skew for the observed or latent variables? Implications for the observed means. Latent skew suitable for GMM - the observed variable means are governed by the growth factor means
Growth Mixture Modeling of NLSY BMI Age 12 to 23 for Black Females ($n = 1160$)

- Normal BIC: 31684 (2c), 31386 (3c), **31314 (4c)**, 31338 (5c)
- Skew-T BIC: 31411 (1c), **31225 (2c)**, 31270 (3c)
2-Class Skew-T versus 4-Class Normal

Class 2, 45.8%
Class 1, 54.2%

Class 1, 18.1%
Class 2, 42.3%
Class 3, 30.1%
Class 4, 9.6%
2-Class Skew-T: Estimated Percentiles
(Note: Not Growth Curves)
2-Class Skew-T: Intercept Growth Factor (Age 17)

Percentiles for I in Class 1 (low class):

- 5%-tile: 18.026
- 10%-tile: 18.656
- 25%-tile: 19.916
- 50%-tile: 21.717
- 75%-tile: 24.327
- 90%-tile: 27.658
- 95%-tile: 30.448

Percentiles for I in Class 2 (high class):

- 5%-tile: 22.169
- 10%-tile: 22.586
- 25%-tile: 23.835
- 50%-tile: 26.333
- 75%-tile: 30.220
- 90%-tile: 35.494
- 95%-tile: 39.936
Regressing Class on a MOMED Covariate ("c ON x"): 2-Class Skew-T versus 4-Class Normal

Recall the estimated trajectory means for skew-t versus normal:
Classic data set
Different age range: 25 to 65
Individually-varying times of observations

Quadratic growth mixture model

Normal distribution BIC is not informative:
- 16557 (1c), 15995 (2c), 15871 (3c), 15730 (4c), 15674 (5c)

Skew-T distribution BIC points to 3 classes:
- 15611 (1c), 15327 (2c), 15296 (3c), 15304 (4c)
Framingham BMI, Females Ages 25 to 65: 3-Class Skew-T

20 30 40 50 60 70

13% 33% 54%

Bengt Muthén
BMI = $kg/m^2$ with normal range 18.5 to 25, overweight 25 to 30, obese > 30

Risk of developing heart disease, high blood pressure, stroke, diabetes

Framingham data contains data on blood pressure treatment at each measurement occasion

Survival component for the first treatment can be added to the growth mixture model with survival as a function of trajectory class
Treatment probabilities are significantly different for the 3 trajectory classes (due to different f intercepts) and in the expected order. Probability plots can be made for each class as a function of age.
Healthy and Retirement Study (HRS) data
Different age range: 65 to death
GMM + non-ignorable dropout as a function of latent trajectory class
Growth Mixture Modeling: Math Achievement Trajectory Classes and High School Dropout. An Example of Substantive Checking via Predictive Validity

Poor Development: 20%  Moderate Development: 28%  Good Development: 52%

Dropout: 69%  8%  1%


- Does the normal mixture solution hold up when checking with non-normal mixtures?
Growth Mixture Modeling: Math Achievement Trajectory Classes and High School Dropout
Growth Mixture Modeling: Math Achievement Trajectory Classes and High School Dropout

Descriptive statistics for MATH10:

- **n = 2040**
- **Mean**: 63.574, **Min**: 29.600
- **Variance**: 186.295, **20%-tile**: 51.430
- **Std dev.:** 13.649, **40%-tile**: 61.810
- **Skewness**: -0.317, **Median**: 65.305
- **Kurtosis**: -0.467, **60%-tile**: 68.400
- **% with Min**: 0.05%, **80%-tile**: 75.280
- **% with Max**: 0.25%, **Max**: 95.170

(1076 missing cases were not included.)
Best solutions, 3 classes (LL, no. par’s, BIC):

- Normal distribution: -34459, 32, 69175
- T distribution: -34453, 35, 69188
- Skew-normal distribution: -34442, 38, 69191
- Skew-t distribution: -34439, 42, 69207

Percent in low, flat class and odds ratios for dropout vs not, comparing low, flat class with the best class:

- Normal distribution: 18 %, OR = 17.1
- T distribution: 19 %, OR = 20.6
- Skew-normal distribution: 26 %, OR = 23.8
- Skew-t distribution: 26 %, OR = 37.3
Cat’s Cradle Concern: A Simulated Case ($n = 2,000$)

Data generated by a 3-class skew-t. 3-class skew-t, BIC=43566:

4-class normal - cat’s cradle with a high/chronic class, BIC=44935:
Disadvantages of Non-Normal Mixture Modeling

- Much slower computations than normal mixtures, especially for large sample sizes
- Needs larger samples; small class sizes can create problems (but successful analyses can be done at $n = 100-200$)
- Needs more random starts than normal mixtures to replicate the best loglikelihood
- Lower entropy
- Needs continuous variables
- Needs continuous variables with many distinct values: Likert scales treated as continuous variables may not carry enough information
- Models requiring numerical integration not yet implemented (required with factors behind categorical and count variables, although maybe not enough information)
Advantages of Non-Normal Mixture Modeling

Non-normal mixtures

- Can fit the data considerably better than normal mixtures
- Can use a more parsimonious model
- Can reduce the risk of extracting latent classes that are merely due to non-normality of the outcomes
- Can check the stability/reproducibility of a normal mixture solution
- Can describe the percentiles of skewed distributions
DISTRIBUTION=SKEW/T/SKEWNORMAL/TDIST in the ANALYSIS command makes it possible to access non-normality parameters in the MODEL command

- Skew parameters are given as \( \{y\} \), \( \{f\} \), where the default is \( \{f\} \) and class-varying. Having both \( \{y\} \) and \( \{f\} \) is not identified.
- Degrees of freedom parameters are given as \( \{df\} \) where the default is class-varying.
- \( df < 1 \): mean not defined, \( df < 2 \): variance not defined, \( df < 3 \): skewness not defined. Density can still be obtained.
- Class-varying \( \{f\} \) makes it natural to specify class-varying \( f \) variance.
- Normal part of the distribution can get zero variances (fixed automatically), with only the non-normal part remaining.
VARIABLE:  
NAMES = id gender age_1996 age_1997 race1 bmi12_2 bmi13_2 bmi14_2 bmi15_2 bmi16_2 bmi17_2 bmi18_2 bmi19_2 bmi20_2 bmi21_2 bmi22_2 bmi23_2 black hisp mixed c1 c2 c3 c1_wom c2_wom c3_wom momedu par bmi bio1_bmi bio2_bmi bmi_par currsmkr97 bingedrnk97 mjuse97 cent_msa liv2prnts adopted income hhs size97;
USEVARIABLES = bmi12_2 bmi13_2 bmi14_2 bmi15_2 bmi16_2 bmi17_2 bmi18_2 bmi19_2 bmi20_2 bmi21_2 bmi22_2 bmi23_2;
USEOBSERVATIONS = gender EQ -1;
MISSING = ALL (9999);
CLASSES = c(2);

ANALYSIS:
TYPE = MIXTURE;
STARTS = 400 80;
PROCESSORS = 8;
DISTRIBUTION = SKEWT;
ESTIMATOR = MLR;
MODEL: %OVERALL%
  i s q | bmi12_2@-.5 bmi13_2@-.4 bmi14_2@-.3
  bmi15_2@-.2 bmi16_2@-.1 bmi17_2@0 bmi18_2@.1
  bmi19_2@.2 bmi20_2@.3 bmi21_2@.4 bmi22_2@.5 bmi23_2@.6;
%c#1%
  i-q;
  i-q WITH i-q;
  bmi12_2-bmi23_2(1);
%c#2%
  i-q;
  i-q WITH i-q;
  bmi12_2-bmi23_2(2);

OUTPUT: TECH1 TECH4 TECH8 RESIDUAL;

PLOT: TYPE = PLOT3;
  SERIES = bmi12_2-bmi23_2(s);
### Skew and Df Parameters

#### Latent Class 1

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<th>Std Err</th>
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<th>P</th>
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<td>0.343</td>
<td>18.175</td>
<td>0.000</td>
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<tr>
<td>S</td>
<td>3.361</td>
<td>0.542</td>
<td>6.204</td>
<td>0.000</td>
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#### Latent Class 2

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<td>-2.296</td>
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<td>Q</td>
<td>3.399</td>
<td>1.281</td>
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<tr>
<td>DF</td>
<td>3.855</td>
<td>0.562</td>
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Technical 4 Output: Estimates derived from the model for Class 1

**Estimated means for the latent variables**

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**Estimated covariance matrix for the latent variables**

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<tr>
<td>S</td>
<td>25.959</td>
<td>40.531</td>
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<tr>
<td>Q</td>
<td>-21.212</td>
<td>32.220</td>
<td>113.186</td>
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## Estimated correlation matrix for the latent variables

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<tbody>
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<tr>
<td>S</td>
<td>0.588</td>
<td>1.000</td>
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<tr>
<td>Q</td>
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<td>0.476</td>
<td>1.000</td>
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## Estimated skew for the latent variables

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<th>S</th>
<th>Q</th>
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<tbody>
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<td>I</td>
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<td></td>
</tr>
<tr>
<td>S</td>
<td>6.653</td>
<td>3.437</td>
<td>-1.588</td>
</tr>
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Non-Normal SEM with Skew-normal, t, and skew-t distributions:

- Allowing a more general model, including non-linear conditional expectation functions
- Chi-square test of model fit
- Percentile estimation of the factor distributions
ML robustness to non-normality doesn’t hold if residuals and factors are not independent or if factors don’t have an unrestricted covariance matrix (Satorra, 2002).

Asparouhov-Muthén (2014):

*There is a preconceived notion that standard structural models are sufficient as long as the standard errors of the parameter estimates are adjusted for failure of the normality assumption, but this is not really correct. Even with robust estimation the data is reduced to means and covariances. Only the standard errors of the parameter estimates extract additional information from the data. The parameter estimates themselves remain the same, i.e., the structural model is still concerned with fitting only the means and the covariances and ignoring everything else.*
\[ Y = \nu + \Lambda \eta + \varepsilon \]
\[ \eta = \alpha + B \eta + \Gamma X + \xi \]

where
\[ (\varepsilon, \xi) \sim rMST(0, \Sigma_0, \delta, DF) \]

and
\[ \Sigma_0 = \begin{pmatrix} \Theta & 0 \\ 0 & \Psi \end{pmatrix} \].

The vector of parameters \( \delta \) is of size \( P + M \) and can be decomposed as \( \delta = (\delta_Y, \delta_\eta) \). From the above equations we obtain the conditional distributions

\[ \eta|X \sim rMST((I - B)^{-1}(\alpha + \Gamma X), (I - B)^{-1}\Psi((I - B)^{-1})^T, (I - B)^{-1}\delta_\eta, DF) \]

\[ Y|\eta \sim rMST(\nu + \Lambda \eta, \Theta, \delta_Y, DF) \]

\[ Y|X \sim rMST(\mu, \Sigma, \delta_2, DF) \]
Adding skew and df parameters to the means, variances, and covariances of the unrestricted H1 model

SEM Likelihood-Ratio Chi-Square Testing

Normal  Non-Normal

\[
\begin{array}{c}
H_0 \\
\text{Automatic} \\
H_1
\end{array} 
\quad 
\begin{array}{c}
H_0 \\
\text{Automatic} \\
H_1
\end{array}
\]
Regression of BMI17 on BMI12: Skew-T vs Normal
Regular indirect and direct effects are not valid

Modeling non-normality and non-linearity needs the more general definitions based on counterfactuals.

The key component of the causal effect definitions, 
\(E[Y(x, M(x^*))|C = c, Z = z]\), can be expressed as follows integrating over the mediator \(M\) (\(C\) is covariate, \(Z\) is moderator, \(X\) is ”cause”):

\[
E[Y(x, M(x^*)) | C = c, Z = z] = \int_{-\infty}^{+\infty} E[Y|C = c, Z = z, X = x, M = m] \times f(M; E[M|C = c, Z = z, X = x^*]) \, dM
\]

Figure 6 of Wall et al. (2012):

Factor distribution can be more parsimoniously specified as skew-t than their mixture of normals. Percentiles are obtained for the estimated distribution.