Bayesian SEM:
A more flexible representation of substantive theory

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Abstract

This paper proposes a new approach to factor analysis and structural equation modeling using Bayesian analysis. The new approach replaces parameter specifications of exact zeros and exact equalities with approximate zeros and equalities based on informative, small-variance priors. It is argued that this produces an analysis that better reflects substantive theories. The proposed Bayesian approach is beneficial in applications where parameters are added to a conventional model such that a non-identified model is obtained if maximum-likelihood estimation is applied. This approach is useful for measurement aspects of latent variable modeling such as with CFA and the measurement part of SEM. Three application areas are studied: Cross-loadings in CFA, residual correlations in CFA, and measurement non-invariance in MIMIC modeling. The approach encompasses three elements: Model testing, model estimation, and model modification. Monte Carlo simulations and real data are analyzed using Mplus.

1 Introduction

This paper proposes a new approach to factor analysis and structural equation modeling using Bayesian analysis. It is argued that current analyses using maximum likelihood (ML) and likelihood-ratio χ^2 testing apply unnecessarily strict models to represent hypotheses derived from substantive theory. This often leads to rejection of the model (see, e.g., Marsh et al, 2009) and a series of model modifications that may capitalize on chance (see, e.g. McCallum, 1992). The hypotheses are reflected in parameters fixed at zero or constrained to be equal to other parameters. Examples include zero cross-loadings and zero residual correlations in factor analysis, absence of direct effects from covariates to factor indicators in structural equation modeling, and multiple-group analysis with measurement invariance.

The new approach is intended to produce an analysis that better reflects substantive theories. It does so by replacing the parameter specification of exact zeros and exact equalities with approximate zeros and equalities. The new approach uses Bayesian analysis to specify informative priors for such parameters. If these parameters were all freed in a conventional analysis, the model would not be identified. The Bayesian analysis, however, identifies the model by substantively-driven small-variance priors. Model testing is carried out using posterior predictive checking which is found to be less sensitive than likelihood-ratio χ^2 testing to ignorable degrees of model misspecification. A side product of the proposed approach is information to modify the model in line with the use of modification indices in ML analysis. ML modification indices inform about model improvement when a single parameter is freed and can lead to a long series of modifications. In contrast, the proposed approach informs about model modification when all parameters are freed and does so in a single-step analysis.

Section 2 presents a brief overview of the Bayesian analysis framework that is used. Sections 4 to 6 present three studies that illustrate the new approach. Each study consists of a real-data example showing the problem, the proposed Bayesian solution for the real-data problem, and simulations showing how well the method works. Study 1 considers factor analysis where cross-loadings make simple structure CFA inadequate. As an example, a re-analysis is made of the classic Holzinger-Swineford mental abilities data, where a simple structure does not fit well by ML CFA standards. Study 2 considers residual correlations in factor analysis which make a factor model inadequate. As an example, the big-five factor model is analyzed using an instrument administered in a British household survey, where the hypothesized five-factor pattern is not well recovered by ML CFA or EFA due to many minor factors. Study 3 considers measurement non-invariance in the form of direct effects from covariates to factor indicators. As an example, responses to an instrument measuring antisocial behavior are related to demographic variables using a U.S. national survey, where extensive differential item functioning distorts comparison of factors across subject groups. All analyses are carried out by Bayesian analysis in Mplus (Muthén & Muthén, 1998-2010) and scripts are available at www.statmodel.com. Section 7 concludes.

2 Bayesian analysis

Frequentist analysis (e.g., maximum likelihood) and Bayesian analysis differ by the former viewing parameters as constants and the latter as variables. Maximum likelihood (ML) finds estimates by maximizing a likelihood computed for the data. Bayes combines prior distributions for parameters with the data likelihood to form posterior distributions for the parameter estimates. The priors can be diffuse (non-informative) or informative where the information may come from previous studies. The posterior provides an estimate in the form of a mean, median, or mode of the posterior distribution.

There are many books on Bayesian analysis and most are quite technical. Gelman et al. (2004) provides a good general statistical description, whereas Lynch (2010) gives a somewhat more introductory account. Press (2003) discusses Bayesian factor analysis. Lee

(2007) gives a discussion from a structural equation modeling perspective. Schafer (1997) gives a statistical discussion from a missing data and multiple imputation perspective, whereas Enders (2010) gives an applied discussion of these same topics. Statistical overview articles include Gelfand et al. (1990) and Casella and George (1992). Overview articles of an applied nature and with a latent variable focus include Scheines et al. (1999), Rupp et al. (2004), and Yuan and MacKinnon (2009).

Bayesian analysis is firmly established in mainstream statistics and its popularity is growing. Part of the reason for the increased use of Bayesian analysis is the success of new computational algorithms referred to as Markov chain Monte Carlo (MCMC) methods. Outside of statistics, however, applications of Bayesian analysis lag behind. One possible reason is that Bayesian analysis is perceived as difficult to do, requiring complex statistical specifications such as those used in the flexible, but technically-oriented general Bayes program WinBUGS. These observations were the background for developing Bayesian analysis in Mplus (Muthén & Muthén, 1998-2010). In Mplus, simple analysis specifications with convenient defaults allow easy access to a rich set of analysis possibilities. Diffuse priors are used as the default with the possibility of specifying informative priors. A range of graphics options are available to easily provide information on estimates, convergence, and model fit. For a technical description of the Mplus implementation, see Asparouhov and Muthén (2010a).

Four key points motivate taking an interest in Bayesian analysis:

1. More can be learned about parameter estimates and model fit
2. Better small-sample performance can be obtained and large-sample theory is not needed
3. Analyses can be made less computationally demanding
4. New types of models can be analyzed

Point 1 is illustrated by parameter estimates that do not have a normal distribution.

ML gives a parameter estimate and its standard error and assumes that the distribution of the parameter estimate is normal based on asymptotic (large-sample) theory. In contrast, Bayes does not rely on large-sample theory and provides the whole distribution not assuming that it is normal. The ML confidence interval $Estimate \pm 1.96 \times SE$ assumes a symmetric distribution, whereas the Bayesian credibility interval based on the percentiles of the posterior allows for a strongly skewed distribution. Bayesian exploration of model fit can be done in a flexible way using Posterior Predictive Checking (PPC; see, e.g., Gelman et al., 1996; Gelman et al., 2004, Chapter 6; Lee, 2007, Chapter 5; Scheines et al., 1999). Any suitable test statistics for the observed data can be compared to statistics based on simulated data obtained via draws of parameter values from the posterior distribution, avoiding statistical assumptions about the distribution of the test statistics.

Point 2 is illustrated by better Bayesian small-sample performance for factor analyses prone to Heywood cases and better performance when a small number of clusters are analyzed in multilevel models. For examples, see Asparouhov and Muthén (2010b).

Point 3 may be of interest for an analyst who is hesitant to move from ML estimation to Bayesian estimation. Many models are computationally cumbersome or impossible using ML, such as with categorical outcomes and many latent variables resulting in many dimensions of numerical integration. Such an analyst may view the Bayesian analysis simply as a computational tool for getting estimates that are analogous to what would have been obtained by ML had it been feasible. This is obtained with diffuse priors, in which case ML and Bayesian results are expected to be close in large samples (Browne & Draper, 2006; p. 505).

Point 4 is exemplified by models with a very large number of parameters or where ML does not provide a natural approach. Examples of the former include image analysis (see, e.g., Green, 1996)) and examples of the latter include random change-point analysis (see, e.g., Dominicus et al., 2008). The Bayesian SEM approach proposed in this paper is a further example of the new type of models that can be analyzed.

2.1 Bayesian estimation

A prior is based on prior beliefs regarding the likely values of a parameter. Data informs about the parameter and modifies the prior into a posterior that gives the Bayesian estimate. This is illustrated in Figure 1 which shows distributions for a prior and a posterior for a parameter, together with the likelihood. The likelihood can be thought of as the distribution of the data given a parameter value. ML finds the parameter value that maximizes the likelihood. In Figure 1 the major portion of the prior distribution has values of the parameter that are lower than those of the likelihood. The posterior is obtained as a compromise between the prior and the likelihood.

Priors can be non-informative or informative. A non-informative prior, also called a diffuse prior, has a large variance. A large variance reflects large uncertainty in the parameter value. With a large prior variance the likelihood contributes relatively more information to the formation of the posterior and the estimate is closer to a maximum-likelihood estimate.

[Figure 1 about here.]

2.1.1 Bayes theorem

Formally, the formation of a posterior draws on Bayes Theorem. Consider the probabilities of events A and B, P(A) and P(B). By elementary probability theory the joint event A and B can be expressed in terms of conditional and marginal probabilities:

$$P(A, B) = P(A|B)\ P(B) = P(B|A)\ P(A). \tag{1}$$

Dividing by P(A) it follows that

$$P(B|A) = \frac{P(A|B)\,P(B)}{P(A)}, \tag{2}$$

which is Bayes Theorem. Applied to modeling, let data take the role of A and the parameter values take the role of B. The posterior can then be expressed symbolically as

$$\begin{aligned} posterior &= parameters|data && (3)\\ &= \frac{data|parameters \times parameters}{data} && (4)\\ &= \frac{likelihood \times prior}{data} && (5)\\ &\propto likelihood \times prior, && (6) \end{aligned}$$

where $\propto$ means proportional to, recognizing that the data do not contain parameters so that this term does not need updating when iteratively finding the posterior.

The prior distribution is the key element of Bayesian analysis. Priors reflect prior beliefs in likely parameter values before collecting new data. These beliefs may come from substantive theory and previous studies of similar populations.

2.1.2 Obtaining the posterior distribution

Bayesian estimation uses Markov Chain Monte Carlo (MCMC) algorithms. The idea behind MCMC is that the conditional distribution of one set of parameters given other sets can be used to make random draws of parameter values, ultimately resulting in an approximation of the joint distribution of all the parameters. For a technical discussion, see, e.g., Gelman et al. (2004). For the technical implementation in Mplus, see Asparohov and Muthén (2010b). Denote by $\boldsymbol{\pi}_i$ a vector of unknowns consisting of parameters, latent variables, and missing observations at iteration i. The vector is divided into several sets,

$\boldsymbol{\pi} = (\boldsymbol{\pi}_{1i}, \boldsymbol{\pi}_{2i}, \ldots, \boldsymbol{\pi}_{Si})'$. For example, in an application without latent variables and missing data, the parameters may be divided into means, intercepts, and slopes in one set and variance and residual variances in another set. Normal priors are commonly used for the first set while inverse-Gamma and inverse-Wishart priors are commonly used for the second set. The conditional distribution for the first set is normal and for the second set inverse-Gamma or inverse-Wishart.

The MCMC sequence of random draws is a follows. Using a set of parameter starting values, new $\boldsymbol{\pi}$ values are obtained by the following steps over $i = 1, 2, \ldots, n$ iterations, in each step drawing from a conditional posterior parameter distribution:

$$Step\ 1: \ \boldsymbol{\pi}_{1,i} | \boldsymbol{\pi}_{2,i-1}, \ldots, \boldsymbol{\pi}_{S,i-1},\ data,\ priors \tag{7}$$

$$Step\ 2: \ \boldsymbol{\pi}_{2,i} | \boldsymbol{\pi}_{1,i}, \boldsymbol{\pi}_{3,i-1}, \ldots, \boldsymbol{\pi}_{S,i-1},\ data,\ priors \tag{8}$$

$$\ldots \tag{9}$$

$$Step\ S: \ \boldsymbol{\pi}_{S,i} | \boldsymbol{\pi}_{1,i}, \ldots, \boldsymbol{\pi}_{S-1,i-1},\ data,\ priors. \tag{10}$$

For the step 1 iteration 1, the parameter values for iteration $i - 1 = 0$ are starting values obtained outside the MCMC process. Step 1 produces values for the parameters of $\boldsymbol{\pi}_1$. In the step 2 iteration 1 those values and the starting values for the other parameters produce values for the parameters of $\boldsymbol{\pi}_2$, and so on up to the step S iteration 1. Iterations $2, \ldots, n$ go through the same steps in the same fashion. Typically, several MCMC chains are used, starting from different starting values and using different random seeds when making the random draws. The chains form independent sequences of iterations and gives an opportunity to monitor convergence.

2.1.3 Assessing convergence

In the analyses of this paper convergence is investigated in the following way. Consider n iterations in m chains, where π_{ij} is the value of parameter π in iteration i of chain j. Define the within- and between-chain variation as

$$\bar{\pi}_{.j} = \frac{1}{n}\sum_{i=1}^{n} \pi_{ij}, \tag{11}$$

$$\bar{\pi}_{..} = \frac{1}{m}\sum_{j=1}^{m} \bar{\pi}_{.j}, \tag{12}$$

$$W = \frac{1}{m}\sum_{j=1}^{m} \frac{1}{n}\sum_{i=1}^{n} (\pi_{ij} - \bar{\pi}_{.j})^2, \tag{13}$$

$$B = \frac{1}{m-1}\sum_{j=1}^{m} (\bar{\pi}_{.j} - \bar{\pi}_{..})^2. \tag{14}$$

Convergence is determined using the Gelman-Rubin convergence diagnostic (Gelman & Rubin (1992); Gelman et al., 2004). This considers the potential scale reduction factor (PSR),

$$PSR = \sqrt{\frac{W+B}{W}}, \tag{15}$$

where a PSR value not much larger than 1 is considered evidence of convergence. Gelman et al. (2004) suggests 1.1, or smaller values for all parameters. This means that convergence is achieved when the between-chain variation is small relative to the within-chain variation. Gelman et al. (2004) use a slightly different definition of their potential scale reduction $\hat{R}$, but the difference relative to (15) is a negligible factor of $n/(n-1)$. It may be the case, however, that PSR convergence observed after n iterations may be negated when using more iterations. Because of this, a longer chain should be run to check that PSR values are close to 1 in a long sequence of iterations.

2.1.4 Model fit

Model fit assessment is possible using Posterior Predictive Checking (PPC) introduced in Gelman, Meng and Stern (1996). With continuous outcomes, PPC as implemented in Mplus builds on the standard likelihood-ratio χ^2 statistic in mean- and covariance-structure modeling. This PPC procedure is described in Scheines et al. (1999) and Asparouhov and Muthén (2010a, b) and is briefly reviewed here. Gelman et al. (2004) presents a more general discussion of PPC, not tied to likelihood-ratio χ^2.

A Posterior Predictive p-value (PPP) of model fit can be obtained via a fit statistic f based on the usual likelihood-ratio χ^2 test of an H_0 model against an unrestricted H_1 model. Low PPP indicates poor fit. Let $f(Y,\ X,\ \boldsymbol{\pi}_i)$ be computed for the data $Y,\ X$ using the parameter values at MCMC iteration i. At iteration i, generate a new data set Y_i^* of synthetic or replicated data of the same sample size as the original data. In this generation the parameter values at iteration i are used. For these replicated data the fit statistic $f(Y_i^*,\ X,\ \boldsymbol{\pi}_i)$ is computed. This data generation and fit statistic computation is repeated over the n iterations, after which PPP is approximated by the proportion of iterations where

$$f(Y,\ X,\ \boldsymbol{\pi}_i) < f(Y_i^*,\ X,\ \boldsymbol{\pi}_i). \tag{16}$$

In the Mplus implementation (Asparouhov & Muthén, 2010a) PPP is computed using every 10^{th} iteration among the iterations used to describe the posterior distribution of parameters. A 95% confidence interval is produced for the difference in the f statistic for the real and replicated data. A positive lower limit is in line with a low PPP and indicates poor fit. An excellent-fitting model is expected to have a PPP value around 0.5 and an f statistic difference of zero falling close to the middle of the confidence interval.

It should be noted that the PPP value does not behave like a p-value for a χ^2 test of model fit (see also Hjort et al., 2006). The type I error is not 5% for a correct model. There is not a theory for how low PPP can be before the model is significantly ill-fitting at a certain level. In this sense, PPP is more akin to a structural equation modeling fit index rather than a χ^2 test. Empirical experience with different models and data has to be established for PPP and some simulation studies are presented here. From these simulations and further ones in Asparouhov and Muthén (2010b), however, the usual approach of using p-values of 0.05 or 0.01 appears reasonable.

3 BSEM: A more flexible SEM approach

A new approach to structural equation modeling based on Bayesian analysis is described below. It is intended to produce an analysis that better reflects the researcher's theories and prior beliefs. It does so by systematically using informative priors for parameters that should not be freely estimated according to the the researcher's theories and prior beliefs. In a frequentist analysis such parameters are typically fixed at zero or are constrained to be equal to other parameters. If these parameters were all freed the model would in fact not be identified. The Bayesian analysis, however, identifies the model by substantively-driven small-variance priors. The proposed approach is referred to as BSEM for Bayesian structural equation modeling. It should be recognized, however, that BSEM refers to the specific Bayesian approach proposed here of using informative, small-variance priors to reflect the researcher's theories and prior beliefs.

BSEM is applicable to any constrained parameter in an SEM. This paper focuses on parameters in the measurement part, but restrictions in the structural part can also be considered. Three types of model features are considered: Cross-loadings in CFA, residual correlations in CFA, and measurement non-invariance.

3.1 Informative priors for cross-loadings in CFA

An analyst who is used to frequentist methods such as ML may at first feel uncomfortable specifying informative priors. It is argued here, however, that a user of CFA is in a sense already engaged in specifying such priors. Consider the confirmatory factor analysis model for an observed p-dimensional vector $\boldsymbol{y}_i$ of factor indicators for individual i,

$$\begin{aligned} \boldsymbol{y}_i &= \boldsymbol{\nu} + \boldsymbol{\Lambda}\, \boldsymbol{\eta}_i + \boldsymbol{\epsilon}_i, \\ E(\boldsymbol{y}_i) &= \boldsymbol{\nu} + \boldsymbol{\Lambda}\, \boldsymbol{\alpha}, \\ V(\boldsymbol{y}_i) &= \boldsymbol{\Lambda}\, \boldsymbol{\Psi}\, \boldsymbol{\Lambda}' + \boldsymbol{\Theta}, \end{aligned} \tag{17}$$

where $\boldsymbol{\nu}$ is an intercept vector, $\boldsymbol{\Lambda}$ a loading matrix, $\boldsymbol{\eta}_i$ is an m-dimensional factor vector, $\boldsymbol{\epsilon}_i$ is a residual vector, $\boldsymbol{\alpha}$ is a factor mean vector, $\boldsymbol{\Psi}$ is a factor covariance matrix, and $\boldsymbol{\Theta}$ is a residual covariance matrix. Here, $\boldsymbol{\epsilon}$ and $\boldsymbol{\eta}$ are assumed normally distributed and uncorrelated.

Drawing on substantive theory, zero cross-loadings in $\boldsymbol{\Lambda}$ are specified for the factor indicators that are hypothesized to not be influenced by certain factors. An exact zero loading can be viewed as a prior distribution that has mean zero and variance zero. A prior that probably more accurately reflects substantive theory uses a mean of zero and a normal distribution with small variance. Figure 2 shows an example where a loading $\lambda \sim N(0, 0.01)$ so that 95% of the loading variation is between -0.2 and $+0.2$. Using standardized variables, a loading of 0.2 is considered a small loading, implying that this prior essentially says that the cross-loading is close zero, but not exactly zero. The prior is strongly informative, but it is not assumed that the parameter is zero.

[Figure 2 about here.]

It is well-known that in frequentist analysis freeing all cross-loadings in a CFA model such as Table 2 leads to a non-identified model because the m^2 restrictions, where m is the number of factors, necessary to eliminate indeterminacies are not present (see, e.g., Hayashi & Marcoulides, 2006). Using small-variance priors for all cross-loadings, however, brings information into the analysis which avoids the non-identification problem. The choice of variance for the prior should correspond to the the researcher's theories and prior beliefs. As stated above, the variance of 0.01 produces a prior where 95% lies between -0.2 and $+0.2$. Other choices are shown in Table 1.

[Table 1 about here.]

A smaller variance may not let cross-loadings escape sufficiently from their zero prior mean, producing a worse Posterior Predictive p-value. A larger variance may let a cross-loading have too large of a probability of having a substantial value. For example, a variance of 0.05 corresponds to 95% lying between -0.44 and $+0.44$, which on a standardized variable scale approaches a major loading size. When the variance is increased, the prior contributes less information so that the model gets closer to being non-identified which causes non-convergence of the MCMC algorithm. It should be noted that the prior variance should be determined in relation to the scale of the observed and latent variables. A prior variance of 0.01 corresponds to small loadings for variables with unit variance, but it corresponds to a smaller loading for an observed variable with variance larger than one. This means that for convenience observed variables may be brought to a common scale either by multiplying them by constants, or standardizing if the model is scale free.

BSEM has an additional advantage. It produces posterior distributions for cross-loadings which can be used in line with modification indices to free parameters for which the credibility interval does not cover zero. Modification indices pertain to freeing only one parameter at a time and a long sequence of model modification is often needed, running the risk of capitalizing on chance (see, e.g., McCallum, 1992). In contrast, the small-variance prior approach provides information on model modification that considers the fixed parameters jointly in a single analysis.

3.2 Informative priors for residual covariances in CFA

An analogous idea can also be used to study residual correlations among factor indicators. In (17), the residual covariance matrix $\boldsymbol{\Theta}$ is commonly assumed to be diagonal. Some residuals may, however, be correlated due to the omission of several minor factors. It is difficult to foresee which residuals should be covaried and freeing all of them leads to a non-identified model in the conventional ML framework. BSEM resolves this.

Instead of assuming a diagonal residual covariance matrix, a more realistic covariance structure model may be expressed as

$$V(\boldsymbol{y}_i) = \boldsymbol{\Lambda}\ \boldsymbol{\Psi}\ \boldsymbol{\Lambda}' + \boldsymbol{\Omega} + \boldsymbol{\Theta}^*, \tag{18}$$

where $\boldsymbol{\Omega}$ is a covariance matrix for the minor factors, not assumed to be diagonal, and $\boldsymbol{\Theta}^*$ is a diagonal covariance matrix. Here, a freely estimated $\boldsymbol{\Omega}$ is not separately identified from $\boldsymbol{\Lambda}$, $\boldsymbol{\Psi}$, and $\boldsymbol{\Theta}^*$. In Bayesian analysis, however, $\boldsymbol{\Omega}$ can be given an informative prior using the inverse-Wishart distribution so that the posterior distribution can be obtained. In this way, the diagonal and off-diagonal elements of $\boldsymbol{\Omega}$ are restricted to small values. This implies that the residual covariance matrix $\boldsymbol{\Omega} + \boldsymbol{\Theta}^*$ contains residual covariances that are allowed to deviate to a small extent from zero means. Sufficiently stringent priors for the off-diagonal elements are needed so that the essential correlations are channeled via $\boldsymbol{\Lambda}\ \boldsymbol{\Psi}\ \boldsymbol{\Lambda}'$. The sums on the diagonal of $\boldsymbol{\Omega} + \boldsymbol{\Theta}^*$ produce the residual variances. The inverse-Wishart distribution is described in the Appendix.

The BSEM approach for residual covariances outlined in connection with (18) will be referred to as Method 1. A more direct method, Method 2, applies an inverse-Wishart prior directly on $\boldsymbol{\Theta}$ in (17). This approach has been discussed in Press (2003; chapter

15). A disadvantage of both Method 1 and Method 2 is that particular residual covariance elements cannot be given their own priors. For example, an analysis may show that some residual covariances should be freely estimated with non-informative priors because they have 95% credibility intervals that do not cover zero. To this aim, Method 3 makes it possible to specify element-specific normal priors for the residual covariances. Mplus allows two different algorithms for Method 3, a random-walk algorithm (Chib & Greenberg, 1988) and a proposal prior algorithm (Asparouhov & Muthén, 2010a).

The choice of inverse-Wishart prior should be made to reflect prior beliefs in the potential magnitude of residual covariances. This is accomplished by using a sufficiently large choice for the degrees of freedom (df) of the inverse-Wishart distribution. To obtain a proper posterior where the marginal mean and variance is defined, $df \geq p+4$ needs to be chosen, where p is the number of variables y. The prior means for the residual covariances can be chosen as zero and the degree of informativeness specified using the df which affects the marginal prior variance via $df - p$. For example, (28) of the Appendix shows that using the inverse-Wishart prior $IW(\boldsymbol{I}, df)$ with $df = p + 6$ gives a prior standard deviation of 0.1, so that two standard deviations below and above the zero mean correspond to the residual covariance range of -0.2 to $+0.2$. The effect of priors is relative to the variances of the ys. For scale-free models, the variables may be standardized before analysis. For larger sample sizes, the prior needs to use a larger df to give the same effect.

Method 1 and 2 both use conjugate priors and generally produce good mixing in the MCMC chain. One advantage of Method 1 over Method 2 is that the prior for the total residual variance is not tied to the prior of the residual covariances because of the fact that the residual covariance has two components that have different priors. Method 2, however, is simpler. Both versions of Method 3, random walk and proposal prior algorithm, are based on the Metropolis-Hastings algorithm and that generally yields somewhat worse mixing performance. The random walk algorithm has difficulty converging or converges very slowly when the variance-covariance matrix has a large number of parameters. However when a large number of parameters have small prior variance the convergence is fast. The proposal prior algorithm generally mixes fast, but it does not work well when the prior variance is very small.

When estimating the above models all the algorithms are applied to extreme conditions with a nearly unidentified model and near zero prior variance. Convergence of the MCMC sequence should be carefully evaluated and automated convergence criteria such as PSR should not be trusted. The models should be estimated with a large number of MCMC iterations, for example, 50000.

3.3 Informative priors for direct effects in MIMIC modeling

BSEM can be extended to include equality constraints. A typical SEM example is multiple-group analysis with measurement invariance. It is common to find small deviations from exact invariance that cause rejection by the ML LRT. Consider the multiple-group model extension of (17) for individual i in group g ($g = 1, 2, \ldots, G$),

$$\begin{aligned} \boldsymbol{y}_{ig} &= \boldsymbol{\nu}_g + \boldsymbol{\Lambda}_g\ \boldsymbol{\eta}_{ig} + \boldsymbol{\epsilon}_{ig}, \\ E(\boldsymbol{y}_{ig}) &= \boldsymbol{\nu}_g + \boldsymbol{\Lambda}_g\ \boldsymbol{\alpha}_g, \\ V(\boldsymbol{y}_{ig}) &= \boldsymbol{\Lambda}_g\ \boldsymbol{\Psi}_g\ \boldsymbol{\Lambda}'_g + \boldsymbol{\Theta}_g. \end{aligned} \tag{19}$$

Bayesian analysis can be used to relax the hypothesis of exact measurement invariance, modifying the model as

$$\boldsymbol{y}_{ig} = \boldsymbol{\nu} + \boldsymbol{\nu}_g + (\boldsymbol{\Lambda} + \boldsymbol{\Lambda}_g)\ \boldsymbol{\eta}_{ig} + \boldsymbol{\epsilon}_{ig}, \tag{20}$$

where for $g = 2, 3, \ldots G$, $\boldsymbol{\nu}_g$ and $\boldsymbol{\Lambda}_g$ are measurement intercept vectors and loading matrices, respectively, that have small-variance priors, representing minor deviations from the intercept vectors and loading matrices of the reference group $g = 1$.

In this paper the special case of intercept non-invariance is studied, assuming invariance of factor loadings and residual covariance matrices. In this case the modeling can be handled by letting the grouping variables be covariates that influence the factors, also referred to as MIMIC modeling (see, e.g., Muthén, 1989). Here, non-invariance is defined as direct effects from covariates to the factor indicators. Unlike with ML, all direct effects can be included and given small-variance priors such as $\sim N(0, 0.01)$.

4 Study 1: Cross-loadings in CFA

4.1 Holzinger-Swineford mental abilities example: ML analysis

The first example uses data from the classic 1939 factor analysis study by Holzinger and Swineford (1939). Twenty-six tests intended to measure a general factor and five specific factors were administered to seventh and eighth grade students in two schools, the Grant-White school ($n = 145$) and the Pasteur school ($n = 156$). Students from the Grant-White school came from homes where the parents were mostly American-born, whereas students from the Pasteur school came largely from working-class parents of whom many were foreign-born and where their native language was used at home.

Factor analyses of these data have been described e.g. by Harman (1976; pp. 123-132) and Gustafsson (2002). Of the 26 tests, nineteen were intended to measure four domains, five measured general deduction, and two were revisions/new test versions. Typically, the last two are not analyzed. Excluding the five general deduction tests, 19 tests measuring four domains are considered here, where the four domains are spatial ability, verbal ability, speed, and memory. The design of the measurement of the four domains by the 19 tests is shown in the factor loading pattern matrix of Table 2. Here, an X denotes a free loading to be estimated and 0 a fixed, zero loading. This corresponds to a simple structure CFA model with variable complexity one, that is, each variable loads on only one factor.

[Table 2 about here.]

Using maximum-likelihood estimation, the model fit using both confirmatory factor analysis (CFA) and exploratory factor analysis (EFA) is reported in Table 3 for both the Grant-White and Pasteur schools. It is seen that the CFA model is rejected by the likelihood-ratio χ^2 test in both samples. Given the rather small sample sizes, one cannot attribute the poor fit to the χ^2 test being overly sensitive to small misspecifications due to a large sample size as is often done. The common fit indices RMSEA and CFI are also not at acceptable levels. In contrast to the CFA, Table 3 shows that the EFA model fits the data well in both schools.

[Table 3 about here.]

Table 4 shows the EFA factor solution for both schools using the Geomin rotation. The Quartimin rotation gives similar results. For a description of these rotations, see, e.g., Asparouhov and Muthén (2009). The table shows that the major loadings of the EFA correspond to the hypothesized four-factor loading pattern (bolded entries). Several of the tests, however, also have cross-loadings on other factors. There are six significant cross-loadings for the Grant-White solution and ten for the Pasteur solution. This explains the poor fit of the CFA model.

The question arises how to go beyond postulating only the number of factors as in EFA and maintain the essence of the hypothesized factor loading pattern without resorting to an exploratory rotation. Cross-loadings need to be allowed, but a model with freely estimated cross-loadings is not identified. The proposed Bayesian solution to this problem is presented next.

[Table 4 about here.]

4.2 Holzinger-Swineford mental abilities example: Bayesian analysis

This section uses data from the Grant-White and Pasteur schools of the Holzinger-Swineford study to illustrate the Section 3.1 BSEM approach with informative cross-loading priors. The factor loading pattern of the four-factor model of Table 2 is used. Table 5 repeats the fit statistics for the ML CFA and EFA and adds the fit statistics for Bayesian analysis using both the original CFA model and the proposed CFA model with informative, small-variance priors for cross-loadings. The cross-loading priors use variances 0.01. Standardized variables are analyzed.

Table 5 shows that the Bayesian analysis of the CFA model with exact zero cross-loadings gives almost zero Posterior Predictive p-values in line with the ML CFA. In contrast, for the proposed Bayesian CFA with cross-loadings model fit is acceptable in that the Posterior Predictive p-value is 0.343 for Grant-White and 0.068 for Pasteur.

The Bayesian estimates can be used as fixed parameters in an ML analysis in order to get the likelihood-ratio test (LRT) value for the Bayes solution. They can be viewed as a measure of fit that can be compared to the ML likelihood-ratio χ^2 values. It is seen in Table 5 that the Bayesian LRT values for the CFA model are close to those of ML χ^2 values. In contrast, the Bayesian LRT values for the model with cross-loadings shows a great improvement, falling halfway between the ML CFA and EFA χ^2 values.

[Table 5 about here.]

The Bayes factor solutions for the two schools are shown in Table 6. It is interesting to compare the Bayes solution to the ML EFA solution of Table 4. The Bayes factor loadings are on the whole somewhat larger than those for ML and there are far fewer significant cross-loadings. For ML, there are six significant cross-loadings for Grant-White and ten for Pasteur, whereof only three appear for both solutions. Because they appear for both schools, a researcher may be tempted to free these three cross-loadings. For Bayes, the Grant-White sample has only two cross-loadings that are significant (have a 95% credibility interval that does not cover zero) and Pasteur has none. Because of the lack of agreement, freeing the two could be capitalizing on chance and is not necessary on behalf of model fit. Bayes clearly gives a simpler pattern than ML EFA for these data.

It should be emphasized that cross-loadings that are found to be important in BSEM (the 95% credibility interval does not cover zero and the cross-loading has strong substantive backing) can be freely estimated while keeping small-variance priors for other cross-loadings. This should improve the results because the small-variance prior results in a too small estimate for such a cross-loading. Monte Carlo simulations show that this gives better estimation.

The ML EFA factor correlations are smaller than the Bayesian factor correlations. The greater extent of cross-loadings in the EFA may contribute to the lower factor correlations in that less correlation among variables need to go through the factors. The Bayesian factor correlations are not excessively high because the factors are expected to correlate to a substantial degree according to theory. Holzinger and Swineford (1939) hypothesized that the variables are all influenced by a deductive factor that in the current model is not partialled out of the four factors.

The choice of cross-loading prior variance should be linked to the researcher's prior beliefs. It could be argued, however, that the choice of a variance of 0.01 resulting in 95% limits of ± 0.20 is not substantially different from a variance of 0.02 resulting in 95% limits of ±0.28; see Table 1. It may therefore be of interest to vary the prior variance to study sensitivity in the results. Increasing the prior variance tends to increase the variability of the estimates and affect the Posterior Predictive p-value. At a certain point of increasing the prior variance, the model is not sufficiently identified and the MCMC algorithm tends to give non-convergence. In the Grant-White data the prior variance of 0.01 (95% limit ±0.20) used for the Table 6 results gives a p-value of 0.343 with factor correlations in the range $0.4 - 0.6$. A prior variance of 0.02 (95% limit ±0.28) gives a p-value of 0.431 with factor correlations around 0.6. For the Pasteur data convergence is not obtained when increasing the prior variance above 0.01. Analyzing the two schools together, yielding a larger sample of $n = 301$, the size of the factor correlations do not change much when varying the prior variance from 0.01 to 0.07. This is in line with findings from Monte Carlo simulations (not shown) where the prior variance is varied for the model used in Section 4.3 with cross-loadings 0.3.

In summary, BSEM provides a simpler model and a model that better fits the researcher's prior beliefs than ML. BSEM provides an approach which is a compromise between that of EFA and CFA. The ML CFA rejects the hypothesized model, presumably because it is too strict. ML EFA does not reject the model, but the model does not match the researcher's prior beliefs because it only postulates the number of factors, not where the large and small loadings should appear. Furthermore, ML EFA provides a solution through a mechanical rotation algorithm, whereas BSEM uses priors to represent the researcher's beliefs.

[Table 6 about here.]

4.3 Cross-loading simulations

This section discusses Monte Carlo simulations of BSEM applied to factor modeling with cross-loadings. The aim is to demonstrate that the proposed approach provides good results for data with known characteristics.

The factor loading pattern of Table 7 is considered where X denotes a major loading and x cross-loadings. The major loadings are all 0.8. The size of the three cross-loadings are varied as 0.0, 0.1, 0.2, and 0.3 in different simulations. The observed and latent variables have unit variances so the loadings are on a standardized scale. A cross-loading of 0.1 is considered to be of little importance, a cross-loading of 0.2 of some importance, and a cross-loading of 0.3 of importance (Cudeck & O'Dell, 1994). The correlations among the three factors are all 0.5. The factor metric is determined by fixing the first loading for each factor at 0.8. Non-informative priors are used for all parameters except for cross-loadings when those are included in the analysis. Sample sizes of $n = 100$, $n = 200$ and $n = 500$ are studied.

A total of 500 replications are used. The reported parameter estimate is the median in the posterior distribution for each parameter. The key result is what frequentists would refer to as the 95% coverage, that is, the proportion of the replications for which the 95% Bayesian credibility interval covers the true parameter value used to generate the data. For cross-loadings it is also of interest to study what corresponds to power in a frequentist setting. This is computed as the proportion of the replications for which the 95% Bayesian credibility interval does not cover zero. Results are reported only for a representative set of parameters or functions of parameters: the major loading of $y2$, the cross-loading for $y6$, the variance for the first factor, and the correlation between the first and second factor.

[Table 7 about here.]

4.3.1 Bayes, non-informative priors

As a first step, consider Bayesian analysis without informative priors and ignoring cross-loadings. The top part of Table 8 shows the summaries of the simulations for $n = 100$, $n = 200$, and $n = 500$ when data have been generated with zero cross-loadings. Here, the analysis is correctly specified and close to 95% coverage is obtained for all free parameters. The cross-loadings are not estimated in this case so that this entry should be ignored. Posterior Predictive p-values for the model fit assessment are 0.036, 0.032, and 0.024, respectively for the three sample sizes, that is, reasonably close to the nominal 5% level. Table 8 shows that Bayesian analysis with non-informative priors works well in this example.

With cross-loadings of 0.1 the bottom part of Table 8 shows the effects of model misspecification in that the coverage is less good. Again, the cross-loadings are not estimated in this case so that this entry should be ignored. Posterior Predictive p-values for the model fit assessment are 0.056, 0.080, and 0.262, respectively for the three sample sizes. This shows limited power to reject the incorrect model. On the other hand, the misspecification is deemed of little importance given the small size of the cross-loadings.

With cross-loadings of 0.2 (not shown) the Posterior Predictive p-value is 0.196 for $n = 100$, 0.474 for $n = 200$, and 0.984 for $n = 500$, showing excellent power at higher sample sizes. With cross-loadings of 0.3 the Posterior Predictive p-value is 0.544 for $n = 100$, 0.944 for $n = 200$, and 1.000 for $n = 500$, showing that the power is excellent when the cross-loading is of an important magnitude.

[Table 8 about here.]

4.3.2 Comparing ML to Bayes

Model fit assessment comparing ML to Bayesian analysis with non-informative priors is shown in Table 17. The correctly specified model with zero cross-loadings shows an inflated ML p-value of 0.172 at $n = 100$. This small-sample bias is well-known for ML χ^2 testing (see, e.g., Scheines et al., 1999). The Posterior Predictive p-value of 0.036 based on the Bayesian analysis does not show the same problem. For the 0.1 size of the cross-loadings, which is deemed of little substantive importance, ML rejects the model 46% of the time at $n = 500$. This reflects the common notion that the ML LRT χ^2 can be oversensitive to small degrees of model misspecification. For the important degree of misspecification with cross-loading 0.3, the ML test is more powerful than Bayes, but the Bayes power is sufficient for sample sizes of at least $n = 200$.

[Table 9 about here.]

Table 10 shows ML model estimation results as a comparison to the Bayesian analysis with non-informative priors presented earlier. For both the correctly specified model with zero cross-loadings and for the misspecified model with cross-loadings 0.1 the ML coverage is close to that of Bayes. The mean-square-error (MSE) is also similar for Bayes and ML. Based on this, there is no reason to prefer one method over the other.

[Table 10 about here.]

4.3.3 Bayes, informative priors

As the next step, the proposed BSEM approach of using Bayesian analysis with informative, small-variance priors for the cross-loadings is applied. Table 11 shows that as compared to Table 8, the coverage remains largely the same for the top part of the table corresponding to the correctly specified analysis with zero cross-loadings. For cross-loadings of 0.1, however, the bottom part of the table shows that coverage has improved by the introduction of informative, small-variance priors for the cross-loadings. The coverage is acceptable also for the cross-loading. The power to detect the cross-loading is, however, small at this low cross-loading magnitude, 0.038, 0.098 and 0.176, respectively for the three sample sizes. The Posterior Predictive p-value is on the low side in all four cases.

It is interesting to compare the coverage results for the four parameters in the case of cross-loadings 0.1 given in the Table 11 for BSEM and in Table 10 for the ML approach. It is seen that ML is outperformed by BSEM by its use of informative priors.

[Table 11 about here.]

Table 12 shows the results of BSEM where data have been generated with larger cross-loadings of 0.2 and 0.3. Here the coverage is also good with the exception of the cross-loadings. For the cross-loadings, however, the focus is on power as shown in the last columns. For a cross-loading of 0.2 a sample size of $n = 500$ is needed to obtain power above 0.8. For a cross-loading of 0.3 a sample size of $n = 200$ is sufficient to obtain power above 0.8. This shows that the approach of using informative, small-variance priors for cross-loadings leads to a successful way to modify the model, allowing free estimation of the indicated cross-loadings.

The point estimates indicate that the key parameter of factor correlation is overestimated. Note, however, that given the power to detect cross-loadings, estimating them freely results in good point estimates for factor correlations.

In summary, the cross-loading simulation study shows that the Bayesian analysis performs well. It also shows that in terms of parameter coverage and for the case of small cross-loadings ML is inferior to BSEM. In terms of model testing, BSEM avoids the small-sample inflation of the ML χ^2 and also avoids the ML χ^2 sensitivity to rejecting a model with an ignorable degree of misspecification.

[Table 12 about here.]

5 Study 2: Residual correlations in CFA

5.1 British Household Panel Study (BHPS) big-five personality example: ML analysis

A second example uses data from the British Household Panel Study (BHPS) of 2005 and 2006. A 15-item, five-factor instrument uses three items to measure each of the "big-five" personality factors: agreeableness, conscientiousness, extraversion, neuroticism, and openness. Each item uses the question "I see myself as someone who . . ." followed by a statement. There are seven response categories ranging from 1 "does not apply" to 7 applies perfectly". A total of $14,021$ subjects are included. The big-five factors are expected a priori to have low correlations and are known to be related to gender and age; see, e.g. Marsh, Nagengast, and Morin (2010). For simplicity, the current analyses hold age constant by considering the subgroup of ages 50-55. This produces a sample of $n = 691$ females and $n = 589$ males.

The item wording and hypothesized loading pattern are shown in Table 13. For all factors except openness, there are two positively-worded items and one negatively-worded item. Marsh, Nagengast, and Morin (2010) suggest that the four negatively-worded items may a priori have correlated residuals (correlated uniquenesses) when applying factor analysis.

[Table 13 about here.]

Using maximum-likelihood estimation, model fit using confirmatory factor analysis (CFA), CFA with correlated uniquenesses (CU) among the negatively-worded items, and exploratory factor analysis (EFA) is reported in Table 14. It is seen that the fit is not acceptable for either of the two CFA models as judged by χ^2 or the two model fit indices. The EFA model is also rejected by χ^2 and only marginally acceptable for males when judged by RMSEA or CFI.

[Table 14 about here.]

An interesting finding is that the EFA solutions for females and males do not fully capture the hypothesized factors. This is the case using the Geomin rotation as well as using Quartimin and Varimax. The Geomin rotation for each gender is shown in Table 15. The bolded entries are loadings that are the largest for the item. Comparing to Table 13 it is seen that only the factors extraversion, neuroticism, and openness are found, not the agreeableness and conscientiousness factors. A possible reason for this is the existence of correlated residuals among the items. As the CFA with CUs model showed, however, allowing residual correlations among the reverse-coded items is not sufficient. It is likely that in addition to the big-five factors the personality instrument measures many minor factors.

The question arises how correlated residuals can be accounted for while maintaining the hypothesized factor loading pattern. A model with all residual correlations freely estimated is not identified. The proposed BSEM solution to this problem is presented next.

[Table 15 about here.]

5.2 British Household Panel Study (BHPS) big-five personality example: Bayesian analysis

This section uses the big-five personality data in the British Household Panel Study (BHPS) to illustrate the BSEM approach of Section 3.2 with an informative prior for the residual covariance matrix. Method 2 is used with an inverse-Wishart prior $IW(\boldsymbol{I}, df)$ with $df = p + 6 = 21$, corresponding to prior means and standard deviations for residual covariances of zero and 0.1, respectively (see Appendix, (28)). Standardized variables are analyzed. Because of high auto-correlation among the MCMC iterations, only every 10^{th} iterations is used with a total of 100,000 iterations to describe the posterior distribution. Informative cross-loading priors are also used with prior distributions $N(0, 0.01)$.

The Posterior Predictive p-values are 0.534 and 0.518, respectively for females and males indicating a good match between the model and the data. For the two samples 17 and 37 residual covariances, respectively, were found significant in the sense of the 95% Bayesian credibility interval not covering zero. The average absolute residual correlation (range) is 0.329 (−0.462 to 0.647) for females and 0.285 (−0.484 to 0.590) for males. For both genders only one residual correlation exceeds 0.5 in absolute value. This suggests that many small residual correlations need to be included in the factor model, as was expected. The fact that these residual correlations are left out in the ML analyses may contribute to the poor ML fit and the poor ML EFA loading pattern recovery.

Table 16 gives the results for the female and male samples. Standardized loadings are presented so that the results can be compared to the ML EFA of Table 15. The hypothesized major loadings are all recovered at substantial values with no significant cross-loadings. The factor correlations are all small to moderate as was expected. The extraversion, neuroticism, and openness factors that were recovered in the ML EFA of Table 15 have lower correlations in the Bayesian solution than in the ML EFA.

In summary, BSEM provides a solution that better fits the researcher's prior beliefs than ML. The ML CFA rejects the hypothesized model, presumably because it is too strict in terms of requiring exactly zero residual covariances. ML EFA does not recover the researcher's hypothesized big-five factor pattern.

[Table 16 about here.]

5.3 Residual correlations simulations

This section discusses Monte Carlo simulations of the BSEM approach to factor modeling with residual correlations. A factor model with 10 variables and two factors is used, where the first five variables load on only the first factor and the second five variable load only on the second factor. The loadings are all 0.8, the factor variances are 1, and the factor correlation is 0.5. The residual variances are 0.36 so that observed variables all have variances 1. Two residual covariances (correlations) are included, one for the first and sixth variables and one for the second and seventh variables. In this way, ignoring the residual covariances in the modeling tends to inflate the factor correlation. An example would be an instrument administered at two time points, where some variables have residuals that are correlated over time. Residual correlations of 0.0, 0.1, and 0.3 are considered together with sample sizes $n = 200$ and $n = 500$. A total of 500 replications are used and the results presented in the format used for the cross-loading simulations. The simulations present results for all three methods discussed in Section 3.2. For Method 1, both a more informative prior with $df = 30$ and a less informative prior with $df = 14$ $(= p + 4)$ is studied. For Method 2, $df = 30$ is used. For Method 3, the normal prior variance is set at 0.001.

5.3.1 Comparing ML to Bayes

As a first step, model testing using ML and Bayes with non-informative priors is compared for both correctly and misspecified models. Table 17 shows that for residual correlations of 0.0, both ML and Bayes give acceptable rejection rates with the correctly specified model. Both ML and Bayes reject the model with ignorable residual correlations of 0.1, although ML is more sensitive to this misspecification. For the larger misspecification with residual correlations of 0.3, both ML and Bayes show sufficient power to reject the model at both $n = 200$ and $n = 500$.

[Table 17 about here.]

Table 18 and Table 19 summarize the simulation results for BSEM using Method 1 with residual correlations 0.1 and 0.3, respectively. The top two panels give results for $df = 30$ and the bottom two panels for $df = 14$.

Table 18 shows good coverage for all parameters except the residual covariance. There is sufficient power to detect the residual covariance of 0.1 at $n = 500$. There is no important difference between using $df = 30$ and $df = 14$, except that the point estimate and the power for the residual covariance is slightly better for the less informative prior with $df = 14$.

Table 19 shows acceptable coverage when using the less informative prior with df = 14, except for the residual correlation. The power to detect the residual correlations is, however, excellent in all cases. For the more informative prior with $df = 30$, the coverage is less good. The key parameter of the factor correlation shows an important overestimation, which is also seen with $df = 14$.

[Table 18 about here.]

[Table 19 about here.]

Table 20 shows the simulation results for BSEM Methods 2 and 3 using residual correlations of 0.3. The 5% reject proportion for the Posterior Predictive p-value is 0 in all cases. For Method 2 the results are very good except for the residual covariance being underestimated and having poor coverage. The power to detect it is, however, excellent. The factor correlation is also somewhat underestimated. Method 2 performs considerable better than Method 1 as seen in Table 19. The Method 3 results are somewhat worse than those of Method 1 for $df = 14$, with poorer performance for the residual covariance and the factor correlation. The power to detect the residual covariance is, however, excellent also for Method 3. It should be noted that Method 3 is the only one of the three methods that can let such a residual covariance be freely estimated, that is, using a non-informative prior. Using a less informative Method 3 prior with a larger variance of 0.01 did not alter the results very much. In summary, Method 2 performs the bests of the three methods and Method 3 the worst for this simulation setting.

[Table 20 about here.]

Table 21 demonstrates that Method 3 works well when the two residual covariances are freely estimated, that is, using non-informative priors. The remaining residual covariances are using the same informative priors as before. Results are shown for $n = 200$ and $n = 500$.

[Table 21 about here.]

6 Study 3: Direct effects in MIMIC modeling

6.1 Antisocial behavior example: ML analysis

As a third example consider Antisocial Behavior (ASB) data taken from the National Longitudinal Survey of Youth (NLSY), sponsored by the Bureau of Labor Statistics. NLSY data for the analysis include 17 antisocial behavior items that were collected in 1980 when respondents were between the ages of 16 and 23. The ASB items assess the frequency of various behaviors during the past year. In these analyses, the ASB items are dichotomized 0/1 with 0 representing never in the last year. The sample analyzed here consists of 7,326 respondents with complete data on the antisocial behavior items and the background variables of age, gender, and ethnicity. The dichotomous responses are modeled using a probit link as in Muthén (1989), applying maximum-likelihood estimation as in IRT.

EFA of the 17 dichotomous ASB items suggests three factors: property offense, person offense, and drug offense. Table 22 shows the item wording and the EFA solution using maximum-likelihood estimation and the Geomin rotation.

[Table 22 about here.]

Because of the heterogeneity of the national sample, it is of interest to study to which extent the ASB items exhibit measurement invariance with respect to the subject groupings provided by the age, gender, and ethnicity variables. Non-invariance is also referred to as differential item functioning (DIF). DIF can be studied using a three-factor CFA model where the factors are regressed on the covariates of age, gender, and ethnicity. DIF is present when a covariate has a direct effect on an item, over and above the indirect effect via the factors (see, e.g., Muthén, 1989).

A search for the need to include direct effects to represent DIF can involve examining modification indices and/or exploring the significance of direct effects by regressing one item at a time on all covariates. Using a relatively untested measurement instrument in a national sample, however, is likely to produce many instances of direct effects (DIF), leading to a long search that may be capitalizing on chance. On the other hand, allowing all direct effects to be freely estimated gives a non-identified model. The proposed Bayesian solution to this problem is presented next.

6.2 Antisocial behavior example: Bayesian analysis

BSEM is applied here to direct effects in the antisocial behavior example. The 17 binary items measure the three factors of property offense, person offense, and drug offense which are related to the covariates age, gender and ethnicity. All direct effects from the covariates to the items are included using normal priors with mean zero and variance 0.04 in line with the discussion in Section 6.3. In addition, cross-loadings are included with normal priors of mean zero and variance 0.01. The major loadings are chosen as follows for each factor, based on EFA. The Property factor is measured by property, shoplift-gt50, con-goods. The Person factor is measured by fight, force-injure. The Drugs factor is measured by pot-solddrug.

Table 23 shows the factor loadings and factor correlations. The top part of Table 24 shows the effects of the covariates on the factors, whereas the bottom part shows the direct effects of the covariates on the factor indicators. The direct effects for which the 95% Bayesian credibility interval does not cover zero are bolded, showing a total of twelve effects. This illustrates the importance of allowing for all possible direct effects using informative, small-variance priors. Modifying the model without any direct effects using modification indices and freeing one parameter at a time leads to a long model modification process. Further research is of interest to see if such an approach is more prone to capitalizing on chance than the BSEM approach.

[Table 23 about here.]

[Table 24 about here.]

6.3 Direct effect simulations

This section discusses Monte Carlo simulations of the BSEM approach to a MIMIC model with direct effects. The same factor loading pattern as in Table 7 is considered, except with no cross-loadings. The correlations among the three factors are all 0.5 and their variances are one. The factor metric is determined by fixing the first loading for each factor at 1.0. The factor model uses a probit specification. The loadings are all 1.0, which is close to the real antisocial behavior data. There is a single binary covariate and there are three direct effects to y1, y6, and y11. The direct effects are 0.4 which corresponds to a small effect size in terms of Cohen's d considering the mean difference in the binary factor indicators. The value 0.4 is chosen as follows. The variance for an underlying conditionally normal latent response variable y^* in the probit factor model is the variance explained by the factor(s) plus a residual variance fixed at unity. For the loadings chosen the variance explained by the factors is 1 and therefore the total variance is 2. With a small effect size defined as approximately 0.3 of a y^* standard deviation, a value of 0.42 is obtained, which is rounded off to 0.4.

Non-informative priors are used for all parameters except for direct effects. The informative prior for each direct effect is taken to be normal with mean zero and variance 0.04 determined as follows. The direct effect of 0.4 is taken to be in the tail of the prior in line with how the cross-loading of 0.2 was in the tail of its prior. Letting the 95% limit be ± 0.4 results in a prior variance of 0.04 by solving for x in $1.96 \times \sqrt(x) = 0.4$.

Sample sizes of $n = 500$, $n = 1000$ and $n = 2000$ are studied. A total of 500 replications are used. The reported parameter estimate is the median in the posterior distribution for each parameter.

Table 25 shows the results. The 95% coverage is close to 0.95 overall with the exception of the direct effect at $n = 500$. The power estimates shown in the last column suggest that there is ample power to detect the direct effect with $n = 2000$, marginal power at

$n = 1000$, and insufficient power at $n = 500$.

The point estimates indicate that the key parameter for the factor regressed on the covariate is overestimated and increasingly so with larger sample size. It should be noted, however, that for larger sample size the direct effect is likely to be detected and when freely estimated leads to good point estimates.

[Table 25 about here.]

7 Conclusions

This paper proposes a new approach to factor analysis and structural equation modeling using Bayesian analysis. The new approach replaces parameter specifications of exact zeros and exact equalities with approximate zeros and equalities based on informative, small-variance priors. It is argued that this produces an analysis that better reflects substantive theories. The proposed Bayesian approach with informative priors, labeled BSEM, is beneficial in applications where if parameters are added to a conventional model, a non-identified model is obtained using ML. The extra model parameters can be viewed as nuisance parameters that based on substantive theory and previous studies are hypothesized to be close to zero although perhaps not exactly zero. This approach is useful for measurement aspects of latent variable modeling such as with CFA and the measurement part of SEM. Three application areas are studied: Cross-loadings in CFA, residual correlations in CFA, and measurement non-invariance in MIMIC modeling. The approach encompasses three elements: Model testing, model estimation, and model modification. The first two are evaluated by Monte Carlo simulation studies, whereas the third warrants further studies. The Monte Carlo and real-data results can be summarized as follows.

7.1 Summary of findings

Model testing uses a Posterior Predictive Probability (PPP) approach that has not previously been investigated this extensively. It is found that PPP works well both for models with only non-informative priors and for the proposed BSEM approach where some parameters have informative priors. PPP is found to perform better than the ML likelihood-ratio χ^2 test at small sample sizes where ML typically inflates χ^2, and is found to be less sensitive than ML to ignorable deviations from the correct model. PPP is found to have sufficient power to detect important model misspecifications.

Bayesian model estimation is shown to perform well with both non-informative and informative priors. Using BSEM with both ignorable and non-ignorable degrees of model misspecification, key parameters are well estimated in terms of their coverage. BSEM outperforms ML estimation with misspecified models.

BSEM also provides a counterpart to ML-based model modification. ML modification indices inform about model improvement when a single parameter is freed and can lead to a long series of modifications. In contrast, BSEM informs about model modification when all parameters are freed and does so in a single step. The simulations show sufficient power to detect model misspecification in terms of 95% Bayesian credibility intervals not covering zero.

An example for each of the three application areas show the promise of BSEM. For the Holzinger-Swineford example a well-fitting factor model is found that is superior to ML-based models. Instead of choosing between an ill-fitting ML CFA model and a well-fitting but unnecessarily weakly specified ML EFA model, BSEM maintains the spirit of CFA while allowing small cross-loadings. For the big-five personality example a well-fitting factor model is found that recovers the hypothesized factor loading pattern by allowing for a large number of small residual correlations. In contrast, ML CFA is ill-fitting even when allowing for a priori residual correlations, and ML EFA does not recover the hypothesized factor loading pattern. For the antisocial behavior example a large number of direct effects from demographic covariates are included to account for measurement non-invariance. In contrast, ML analysis requires a long sequence of model modification to fully recover the non-invariance.

Applying BSEM is easy and fast for analyses of cross-loadings and direct effects. Analysis with residual covariances leads to heavier computations due to slow mixing, producing slow MCMC convergence. A further benefit of the Bayesian analysis is that estimation works well also for models that are large relative to the sample size (see also Asparouhov & Muthén, 2010b).

7.2 Related approaches

BSEM with its adjoining PPP model test is similar in spirit to the frequentist conceptualization of "close fit" (Browne & Cudeck, 1993). ML model testing of close fit rather than conventional exact χ^2 fit is expressed by the root mean square error of approximation (RMSEA) fit index. In assessing differences between models, McCallum et al. (2006) also argue against exact fit as being of limited empirical interest given that it is never true in practice. RMSEA uses an overall approximate fit level deemed sufficient. In contrast, BSEM allows informative priors to reflect notions of closeness for each parameter.

Press (2003; chapter 15) discusses a Bayesian factor analysis approach that has some similarities to the one proposed in this paper. The MCMC algorithm is not used but instead estimates are obtained as expected values in the posterior distributions. Press (2003) specifies a prior for the loading matrix with a mean that uses a specific "target" pattern of large and zero loadings. All loadings have the same prior variances. In the example (Press, 2003; pp. 368-372) the variances are chosen to give weakly informative priors. In contrast, the current approach has zero prior means for all loadings, with small prior variances for non-target loadings and large prior variances for target loadings so that target loadings are solely determined by the data. In this sense, the Press (2003) approach is closer to EFA and the current approach is closer to CFA.

It is of interest to compare BSEM for the case of cross-loadings to the frequentist approach of exploratory factor analysis using target rotation (Brown, 2001; Asparouhov & Muthén, 2009). Target rotation is an EFA technique where a rotation is chosen to match certain zero target loadings using a least-squares fitting function. It is similar to BSEM in that it replaces mechanical rotation with rotation guided by the researcher's judgement, in this case using zero targets for cross-loadings. It is also similar in that the fitting function can result in non-zero values for the targets. It is different from BSEM by not allowing user-specified stringency of closeness to zero by varying the prior variance, replacing that with least-square fitting. It is also different from BSEM because specifying $m-1$ zeros for each of the m factors gives the same model fit as specifying more zeros. For the Holzinger-Swineford example, applying target rotation with zero targets for all cross-loadings gives results similar to EFA using Geomin or Quartimin rotation. The same six cross-loadings are obtained for Grant-White while Pasteur obtains two more cross-loadings, adding to twelve. Again, BSEM using small-variance cross-loading priors gives far simpler loading patterns.

In BSEM the ability to free all loadings in a measurement model can be viewed as the ability to form an EFA with the rotation guided by the priors. BSEM is, however, more general than EFA and essentially has the flexibility of ESEM (Asparouhov & Muthén, 2009) because it can accommodate correlated residuals in an EFA model, it can accommodate covariates in an EFA model, and it can accommodate an EFA model as part of a larger model. BSEM also generalizes ESEM in the following way. In ESEM the optimal rotation is determined based only on the unrotated loadings as in EFA, that is, the optimal rotation does not consider residual covariances or covariate direct effects in the optimal rotations. In contrast, in BSEM the optimal rotation is determined by all parts of the model.

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8 Appendix

The density of the inverse-Wishart distribution $IW(\boldsymbol{S}, d)$ with d degrees of freedom is given by

$$\frac{|\boldsymbol{S}|^{d/2}|X|^{-(d+p+1)/2}Exp(-Tr(\boldsymbol{S}X^{-1})/2)}{2^{dp/2}\Gamma_p(d/2)}, \tag{21}$$

where Γ_p is the multivariate gamma function and the argument X of the density is a positive density function. To use an informative prior with a certain expected value one can use the fact that the mean of the distribution is

$$\frac{\boldsymbol{S}}{d-p-1}. \tag{22}$$

The mean exist and is finite only if $d > p+1$. If $d \leq p+1$ then one can use the fact that the mode of the distribution is

$$\frac{\boldsymbol{S}}{d+p+1}. \tag{23}$$

The variance, i.e., the level of informativeness is controlled exclusively by the parameter d. The larger the value of d the more informative the prior is.

To evaluate the informativeness of the prior one can consider the marginal distribution of the diagonal elements. The marginal distribution of the $j-$th diagonal entry is

$$IG((d-p+1)/2, \boldsymbol{S}_{jj}/2). \tag{24}$$

Thus the marginal mean is

$$\frac{\boldsymbol{S}_{jj}}{d-p-1} \tag{25}$$

if $d > p+1$ and the marginal variance is

$$\frac{2\boldsymbol{S}_{jj}^2}{(d-p-1)^2(d-p-3)} \tag{26}$$

if $d > p + 3$. To use an informative prior with a certain variance one can multiply the desired expected value by $(d - p - 1)$ to get $\boldsymbol{S}$.

The marginal distribution of the off-diagonal elements can not be expressed in closed form, but the marginal mean for the (i, j) off-diagonal element is

$$\frac{\boldsymbol{S}_{ij}}{d - p - 1} \tag{27}$$

if $d > p + 1$ and the marginal variance is

$$\frac{(d - p + 1)\boldsymbol{S}_{ij}^2 + (d - p - 1)\boldsymbol{S}_{ii}\boldsymbol{S}_{jj}}{(d - p)(d - p - 1)^2(d - p - 3)} \tag{28}$$

if $d > p + 3$. As an example, using an identity matrix $\boldsymbol{S} = \boldsymbol{I}$ and $d = p + 6$ for $IW(\boldsymbol{S}, d)$ gives mean zero and variance = 0.0111 (standard deviation = 0.1054).

It is clear that stating the level of informativeness using inverse-Wishart priors is rigid as the informativeness of one parameter in the matrix determines the informativeness of all other parameters. A special case is of particular interest. Setting the prior to $IW(\boldsymbol{D}, p+1)$ where $\boldsymbol{D}$ is a diagonal matrix, the marginal distribution for all correlations is uniform on the interval $(-1, 1)$ while the marginal distributions of the variance is $IG(1, d_{jj}/2)$. The values of the diagonal elements d_{jj} can be set to match the mode of the desired prior with the mode of $IG(1, d_{jj}/2)$ which is $d_{jj}/4$. Note however that the mean can not be used for this purpose since the mean of $IG(1, d_{jj}/2)$ is infinity. Only the mode is defined for this distribution. In this case the marginal distribution of the diagonal elements has infinite mean and variance. The marginal for the covariance elements has mean zero by symmetry but also has an infinite variance. The marginal mean for the correlation parameter is zero and the marginal variance for the correlation parameter is 1/3.

More generally setting the prior to $IW(\boldsymbol{D}, d)$ where $\boldsymbol{D}$ is a diagonal matrix, the marginal distribution for all correlations is the beta distribution $B((d-p+1)/2, (d-p+1)/2)$

on the interval (-1,1), if $d \geq p$ with mean 0 and variance

$$\frac{1}{d-p+2}. \tag{29}$$

Note also that the posterior distribution in the MCMC generation for the variance covariance parameter with prior $IW(\boldsymbol{S}, d)$ is a weighted average of $\boldsymbol{S}/d$ and the sample variance where the weights are $d/(n+d)$ and $n/(n+d)$ respectively, where n is the sample size.. Thus one can interpret the degrees of freedom parameter d as the number of observations added to the sample with the prior variance covariance matrix. Naturally as the sample size increases the weight $d/(n+d)$ will converge to 0 and the effect of the prior matrix $\boldsymbol{S}$ will diminish. To maintain the same effect of the prior on the estimation for larger sample sizes the degrees of freedom parameter should be chosen proportionally larger.

More information on the inverse-Wishart distribution and the marginal distributions of all the entries in the matrix can be found in Barnard et al. (2000).

List of Figures

Figure 1: Prior, likelihood, and posterior for a parameter

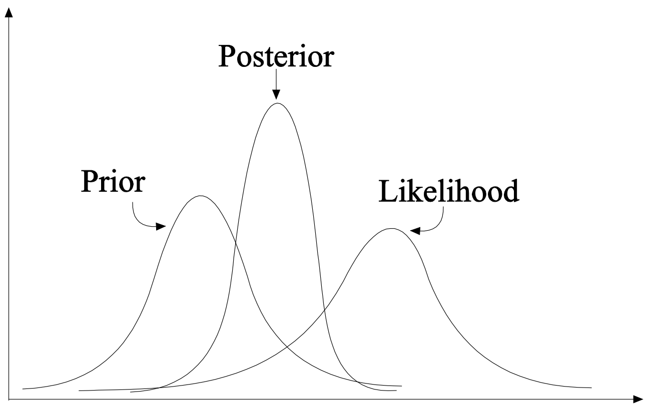

Figure 2: Informative prior for a factor loading parameter

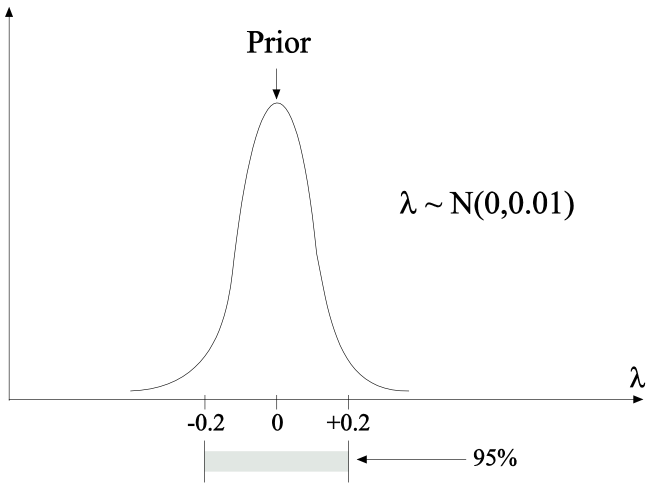

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Table 1: Choice of variance for a normal prior with mean zero

Variance	90% limits	95% limits
0.001	± 0.05	± 0.06
0.005	± 0.12	± 0.14
0.01	± 0.16	± 0.20
0.02	± 0.23	± 0.28
0.03	± 0.28	± 0.34
0.04	± 0.33	± 0.39
0.05	± 0.37	± 0.44
0.1	± 0.52	± 0.62

Table 2: Holzinger-Swineford's hypothesized four domains measured by 19 tests

Factor loading pattern

	Spatial	Verbal	Speed	Memory
visual	X	0	0	0
cubes	X	0	0	0
paper	X	0	0	0
flags	X	0	0	0
general	0	X	0	0
paragrap	0	X	0	0
sentence	0	X	0	0
wordc	0	X	0	0
wordm	0	X	0	0
addition	0	0	X	0
code	0	0	X	0
counting	0	0	X	0
straight	0	0	X	0
wordr	0	0	0	X
numberr	0	0	0	X
figurer	0	0	0	X
object	0	0	0	X
numberf	0	0	0	X
figurew	0	0	0	X

Table 3: ML model testing results for Holzinger-Swineford data for Grant-White ($n = 145$) and Pasteur ($n = 156$)

Model	χ^2	df	p-value	RMSEA	CFI
Grant-White					
CFA	216	146	0.000	0.057	0.930
EFA	110	101	0.248	0.025	0.991
Pasteur					
CFA	261	146	0.000	0.071	0.882
EFA	128	101	0.036	0.041	0.972

Table 4: Holzinger-Swineford ML EFA using 19 variables and Geomin rotation: Four-factor solution

Loadings

	Grant-White				Pasteur			
	Spatial	Verbal	Speed	Memory	Spatial	Verbal	Speed	Memory
visual	**0.628***	0.065	0.091	0.085	**0.580***	0.307*	-0.001	0.053
cubes	**0.485***	0.050	0.007	-0.003	**0.521***	0.027	-0.078	-0.059
paper	**0.406***	0.107	0.084	0.083	**0.484***	0.101	-0.016	-0.229*
flags	**0.579***	0.160	0.013	0.026	**0.687***	-0.051	0.067	0.101
general	0.042	**0.752***	0.126	-0.051	-0.043	**0.838***	0.042	-0.118
paragrap	0.021	**0.804***	-0.056	0.098	0.026	**0.800***	-0.006	0.069
sentence	-0.039	**0.844***	0.085	-0.057	-0.045	**0.911***	-0.054	-0.029
wordc	0.094	**0.556***	0.197*	0.019	0.098	**0.695***	0.008	0.083
wordm	0.004	**0.852***	-0.074	0.069	0.143*	**0.793***	0.029	-0.023
addition	-0.302*	0.029	**0.824***	0.078	-0.247*	0.067	**0.664***	0.026
code	0.012	0.050	**0.479***	0.279*	0.004	0.262*	**0.552***	0.082
counting	0.045	-0.159	**0.826***	-0.014	0.073	-0.034	**0.656***	-0.166
straight	0.346*	0.043	**0.570***	-0.055	0.266*	-0.034	**0.526***	-0.056
wordr	-0.024	0.117	-0.020	**0.523***	-0.005	0.020	-0.039	**0.726***
numberr	0.069	0.021	-0.026	**0.515***	-0.026	-0.057	-0.057	**0.604***
figurer	0.354*	-0.033	-0.077	**0.515***	0.329*	0.042	0.168	**0.403***
object	-0.195	0.045	0.154	**0.685***	-0.123	-0.005	0.333*	**0.469***
numberf	0.225	-0.127	0.246*	**0.450***	0.246*	0.092	0.092	**0.427***
figurew	0.069	0.099	0.058	**0.365***	0.139	0.013	0.237*	**0.291***

Factor Correlations

	Grant-White				Pasteur			
	Spatial	Verbal	Speed	Memory	Spatial	Verbal	Speed	Memory
Spatial	1.000				1.000			
Verbal	0.378*	1.000			0.186*	1.000		
Speed	0.372*	0.386*	1.000		0.214	0.326*	1.000	
Memory	0.307*	0.380*	0.375*	1.000	0.190*	0.100	0.242*	1.000

Table 5: ML versus Bayes model testing results for Holzinger-Swineford data for Grant-White ($n = 145$) and Pasteur ($n = 156$)

ML analysis					
Model	χ^2	df	p-value	RMSEA	CFI
Grant-White					
CFA	216	146	0.000	0.057	0.930
EFA	110	101	0.248	0.025	0.991
Pasteur					
CFA	261	146	0.000	0.071	0.882
EFA	128	101	0.036	0.041	0.972

Bayesian analysis				
Model	Sample LRT	2.5% PP limit	97.5% PP limit	PP p-value
Grant-White				
CFA	219	12	112	0.006
CFA w. cross-loadings	160	-31	69	0.242
Pasteur				
CFA	264	56	156	0.000
CFA w. cross-loadings	175	-13	90	0.068

Table 6: Bayes for Holzinger-Swineford example: Four-factor solution using informative priors for cross-loadings

Loadings

	Grant-White				Pasteur			
	Spatial	Verbal	Speed	Memory	Spatial	Verbal	Speed	Memory
visual	**0.762***	-0.022	0.011	0.002	**0.710***	0.109	0.015	0.010
cubes	**0.507***	-0.008	-0.014	-0.005	**0.479***	-0.029	-0.027	-0.020
paper	**0.489***	0.028	0.032	0.014	**0.505***	0.010	-0.023	-0.080
flags	**0.667***	0.035	-0.024	0.000	**0.671***	-0.087	0.017	0.062
general	0.028	**0.778***	0.043	-0.014	-0.052	**0.855***	0.018	-0.061
paragrap	0.003	**0.832***	-0.033	0.014	0.022	**0.796***	-0.002	0.036
sentence	-0.041	**0.867***	0.013	-0.022	-0.058	**0.917***	-0.021	-0.030
wordc	0.048	**0.624***	0.082	0.017	0.046	**0.698***	0.012	0.051
wordm	-0.009	**0.879***	-0.065	0.010	0.074	**0.806***	0.002	0.019
addition	-0.134	0.021	**0.782***	0.000	-0.100	-0.010	**0.656***	-0.001
code	0.016	0.052	**0.601***	0.049	0.008	0.101	**0.665***	0.035
counting	0.009	-0.075	**0.796***	-0.018	0.015	-0.044	**0.597***	-0.034
straight	0.156	0.042	**0.629***	-0.012	0.098	-0.053	**0.557***	0.006
wordr	-0.027	0.042	-0.010	**0.516***	-0.040	-0.001	-0.052	**0.691***
numberr	0.001	-0.002	-0.030	**0.543***	0.005	-0.093	-0.067	**0.610***
figurer	0.102	-0.010	-0.034	**0.560***	0.112	0.039	0.042	**0.539***
object	-0.105	0.010	0.021	**0.713***	-0.067	0.015	0.077	**0.550***
numberf	0.074	-0.052	0.067	**0.584***	-0.016	0.039	0.000	**0.471***
figurew	0.015	0.037	0.003	**0.448***	0.032	0.029	0.053	**0.417***

Factor Correlations

	Grant-White				Pasteur			
	Spatial	Verbal	Speed	Memory	Spatial	Verbal	Speed	Memory
Spatial	1.000				1.000			
Verbal	0.548*	1.000			0.373*	1.000		
Speed	0.500*	0.430*	1.000		0.304*	0.445*	1.000	
Memory	0.557*	0.494*	0.557*	1.000	0.341*	0.163	0.400*	1.000

Table 7: Factor loading pattern for simulation study of cross-loadings

Factor loading pattern

	F1	F2	F3
y1	X	0	x
y2	X	0	0
y3	X	0	0
y4	X	0	0
y5	X	0	0
y6	x	X	0
y7	0	X	0
y8	0	X	0
y9	0	X	0
y10	0	X	0
y11	0	x	X
y12	0	0	X
y13	0	0	X
y14	0	0	X
y15	0	0	X

Table 8: Bayesian analysis, non-informative cross-loading priors, 0.0 and 0.1 cross-loadings

Parameter	Estimates			S.E.	M.S.E.	95%	% Sig
	Population	Average	Std. Dev.	Average		Cover	Coeff
Cross-loading = 0.0, n=100, 5% reject proportion for the PPP = 0.036							
Major loading	0.800	0.8192	0.1034	0.1005	0.0110	0.928	1.000
Cross-loading	0.000	0.0000	0.0000	0.0000	0.0000	1.000	0.000
Factor variance	1.000	1.0475	0.2447	0.2365	0.0620	0.938	1.000
Factor correlation	0.500	0.5026	0.0872	0.0886	0.0076	0.952	0.996
Cross-loading = 0.0, n=200, 5% reject proportion for the PPP = 0.032							
Major loading	0.800	0.8114	0.0680	0.0677	0.0047	0.948	1.000
Cross-loading	0.000	0.0000	0.0000	0.0000	0.0000	1.000	0.000
Factor variance	1.000	1.0114	0.1635	0.1564	0.0268	0.922	1.000
Factor correlation	0.500	0.4999	0.0602	0.0621	0.0036	0.954	1.000
Cross-loading = 0.0, n=500, 5% reject proportion for the PPP = 0.024							
Major loading	0.800	0.8049	0.0412	0.0415	0.0017	0.946	1.000
Cross-loading	0.000	0.0000	0.0000	0.0000	0.0000	1.000	0.000
Factor variance	1.000	1.0024	0.0974	0.0960	0.0095	0.940	1.000
Factor correlation	0.500	0.4999	0.0379	0.0390	0.0014	0.954	1.000
Cross-loading = 0.1, n=100, 5% reject proportion for the PPP = 0.056							
Major loading	0.800	0.7605	0.0934	0.0905	0.0103	0.902	1.000
Cross-loading	0.000	0.0000	0.0000	0.0000	0.0000	1.000	0.000
Factor variance	1.000	1.2063	0.2713	0.2631	0.1160	0.832	1.000
Factor correlation	0.500	0.5278	0.0843	0.0859	0.0079	0.940	0.996
Cross-loading = 0.1, n=200, 5% reject proportion for the PPP = 0.080							
Major loading	0.800	0.7537	0.0615	0.0612	0.0059	0.860	1.000
Cross-loading	0.000	0.0000	0.0000	0.0000	0.0000	1.000	0.000
Factor variance	1.000	1.1645	0.1796	0.1731	0.0592	0.806	1.000
Factor correlation	0.500	0.5256	0.0584	0.0600	0.0041	0.932	1.000
Cross-loading = 0.1, n=500, 5% reject proportion for the PPP = 0.262							
Major loading	0.800	0.7496	0.0386	0.0374	0.0040	0.700	1.000
Cross-loading	0.000	0.0000	0.0000	0.0000	0.0000	1.000	0.000
Factor variance	1.000	1.1498	0.1092	0.1062	0.0343	0.684	1.000
Factor correlation	0.500	0.5264	0.0365	0.0376	0.0020	0.906	1.000

Table 9: Rejection rates for ML CFA and Bayes CFA with non-informative priors

Cross-loading	Sample size	ML LRT rejection rate	Bayes PPP rejection rate
0.0	100	0.172	0.036
	200	0.090	0.032
	500	0.060	0.024
0.1	100	0.226	0.056
	200	0.228	0.080
	500	0.460	0.262
0.2	100	0.488	0.196
	200	0.726	0.474
	500	0.998	0.984
0.3	100	0.830	0.544
	200	0.996	0.944
	500	1.000	1.000

Table 10: ML: cross-loadings 0.0 and 0.1

Parameter	Estimates			S.E.	M.S.E.	95%	% Sig
	Population	Average	Std. Dev.	Average		Cover	Coeff
Cross-loading = 0.0, n=100, 5% reject proportion for the LRT χ^2 test = 0.172							
Major loading	0.800	0.8066	0.0984	0.0945	0.0097	0.944	1.000
Cross-loading	0.000	0.0000	0.0000	0.0000	0.0000	1.000	0.000
Factor variance	1.000	0.9842	0.2079	0.2091	0.0435	0.932	1.000
Factor correlation	0.500	0.4926	0.0886	0.0867	0.0079	0.936	0.996
Cross-loading = 0.0, n=200, 5% reject proportion for the LRT χ^2 test = 0.090							
Major loading	0.800	0.8007	0.0667	0.0659	0.0044	0.954	1.000
Cross-loading	0.000	0.0000	0.0000	0.0000	0.0000	1.000	0.000
Factor variance	1.000	0.9870	0.1476	0.1495	0.0219	0.952	1.000
Factor correlation	0.500	0.4949	0.0632	0.0613	0.0040	0.940	1.000
Cross-loading = 0.0, n=500, 5% reject proportion for the LRT χ^2 test = 0.060							
Major loading	0.800	0.8017	0.0420	0.0413	0.0018	0.942	1.000
Cross-loading	0.000	0.0000	0.0000	0.0000	0.0000	1.000	0.000
Factor variance	1.000	0.9970	0.0948	0.0954	0.0090	0.950	1.000
Factor correlation	0.500	0.4978	0.0373	0.0387	0.0014	0.958	1.000
Cross-loading = 0.1, n=100, 5% reject proportion for the LRT χ^2 test = 0.226							
Major loading	0.800	0.7510	0.0894	0.0852	0.0104	0.884	1.000
Cross-loading	0.000	0.0000	0.0000	0.0000	0.0000	1.000	0.000
Factor variance	1.000	1.1281	0.2373	0.2318	0.0726	0.944	1.000
Factor correlation	0.500	0.5192	0.0863	0.0839	0.0078	0.926	0.996
Cross-loading = 0.1, n=200, 5% reject proportion for the LRT χ^2 test = 0.228							
Major loading	0.800	0.7458	0.0606	0.0595	0.0066	0.834	1.000
Cross-loading	0.100	0.0000	0.0000	0.0000	0.0000	1.000	0.000
Factor variance	1.000	1.1316	0.1638	0.1649	0.0441	0.906	1.000
Factor correlation	0.500	0.5213	0.0616	0.0593	0.0042	0.922	1.000
Cross-loading = 0.1, n=500, 5% reject proportion for the LRT χ^2 test = 0.460							
Major loading	0.800	0.7470	0.0381	0.0373	0.0043	0.662	1.000
Cross-loading	0.000	0.0000	0.0000	0.0000	0.0000	1.000	0.000
Factor variance	1.000	1.1430	0.1049	0.1052	0.0314	0.744	1.000
Factor correlation	0.500	0.5242	0.0362	0.0375	0.0019	0.892	1.000

Table 11: Bayesian analysis using informative, small-variance priors for cross-loadings 0.0 and 0.1

Parameter	Estimates			S.E.	M.S.E.	95%	% Sig
	Population	Average	Std. Dev.	Average		Cover	Coeff
Cross-loading = 0.0, n=100, 5% reject proportion for the PPP = 0.006							
Major loading	0.800	0.8472	0.1212	0.1310	0.0169	0.950	1.000
Cross-loading	0.000	0.0141	0.0455	0.0732	0.0023	0.998	0.002
Factor variance	1.000	0.9864	0.2595	0.2624	0.0674	0.930	1.000
Factor correlation	0.500	0.4967	0.0869	0.1076	0.0075	0.980	0.982
Cross-loading = 0.0, n=200, 5% reject proportion for the PPP = 0.002							
Major loading	0.800	0.8311	0.0893	0.0894	0.0089	0.942	1.000
Cross-loading	0.000	0.0079	0.0421	0.0633	0.0018	1.000	0.000
Factor variance	1.000	0.9662	0.1799	0.1840	0.0335	0.948	1.000
Factor correlation	0.500	0.4962	0.0605	0.0860	0.0037	0.990	1.000
Cross-loading = 0.0, n=500, 5% reject proportion for the PPP = 0.010							
Major loading	0.800	0.8161	0.0520	0.0552	0.0030	0.962	1.000
Cross-loading	0.000	0.0033	0.0313	0.0530	0.0010	1.000	0.000
Factor variance	1.000	0.9741	0.1096	0.1293	0.0127	0.958	1.000
Factor correlation	0.500	0.4960	0.0406	0.0708	0.0017	1.000	1.000
Cross-loading = 0.1, n=100, 5% reject proportion for the PPP = 0.006							
Major loading	0.800	0.8218	0.1177	0.1263	0.0143	0.950	1.000
Cross-loading	0.100	0.0594	0.0449	0.0728	0.0037	0.982	0.038
Factor variance	1.000	1.0600	0.2738	0.2808	0.0784	0.934	1.000
Factor correlation	0.500	0.5206	0.0850	0.1047	0.0076	0.976	0.992
Cross-loading = 0.1, n=200, 5% reject proportion for the PPP = 0.006							
Major loading	0.800	0.8151	0.0860	0.0882	0.0076	0.950	1.000
Cross-loading	0.000	0.0666	0.0424	0.0636	0.0029	0.978	0.098
Factor variance	1.000	1.0217	0.1895	0.1978	0.0363	0.942	1.000
Factor correlation	0.500	0.5204	0.0602	0.0843	0.0040	0.984	1.000
Cross-loading = 0.1, n=500, 5% reject proportion for the PPP = 0.008							
Major loading	0.800	0.8089	0.0517	0.0551	0.0027	0.964	1.000
Cross-loading	0.100	0.0732	0.0316	0.0532	0.0017	0.990	0.176
Factor variance	1.000	1.0169	0.1160	0.1371	0.0137	0.972	1.000
Factor correlation	0.500	0.5229	0.0404	0.0688	0.0022	0.998	1.000

Table 12: Bayesian analysis using informative, small-variance priors for cross-loadings 0.2 and 0.3

Parameter	Estimates			S.E.	M.S.E.	95%	% Sig
	Population	Average	Std. Dev.	Average		Cover	Coeff
Cross-loading = 0.2, n=100, 5% reject proportion for the PPP = 0.010							
Major loading	0.800	0.7952	0.1152	0.1217	0.0133	0.952	1.000
Cross-loading	0.200	0.1024	0.0453	0.0731	0.0116	0.840	0.188
Factor variance	1.000	1.1522	0.2990	0.3018	0.1124	0.908	1.000
Factor correlation	0.500	0.5439	0.0819	0.1023	0.0086	0.966	0.996
Cross-loading = 0.2, n=200, 5% reject proportion for the PPP = 0.004							
Major loading	0.800	0.7979	0.0835	0.0859	0.0070	0.940	1.000
Cross-loading	0.200	0.1239	0.0418	0.0638	0.0075	0.856	0.492
Factor variance	1.000	1.0850	0.1978	0.2109	0.0463	0.942	1.000
Factor correlation	0.500	0.5424	0.0581	0.0823	0.0052	0.974	1.000
Cross-loading = 0.2, n=500, 5% reject proportion for the PPP = 0.006							
Major loading	0.800	0.8010	0.0514	0.0554	0.0026	0.964	1.000
Cross-loading	0.200	0.1427	0.0327	0.0541	0.0044	0.922	0.854
Factor variance	1.000	1.0595	0.1222	0.1473	0.0184	0.966	1.000
Factor correlation	0.500	0.5445	0.0382	0.0669	0.0034	0.986	1.000
Cross-loading = 0.3, n=100, 5% reject proportion for the PPP = 0.012							
Major loading	0.800	0.7671	0.1104	0.1166	0.0139	0.924	1.000
Cross-loading	0.300	0.1428	0.0460	0.0734	0.0268	0.364	0.470
Factor variance	1.000	1.2532	0.3158	0.3281	0.1636	0.860	1.000
Factor correlation	0.500	0.5650	0.0810	0.0999	0.0108	0.952	0.996
Cross-loading = 0.3, n=200, 5% reject proportion for the PPP = 0.012							
Major loading	0.800	0.7790	0.0807	0.0836	0.0069	0.948	1.000
Cross-loading	0.300	0.1755	0.0419	0.0640	0.0173	0.518	0.856
Factor variance	1.000	1.1623	0.2134	0.2257	0.0718	0.890	1.000
Factor correlation	0.500	0.5642	0.0577	0.0804	0.0075	0.950	1.000
Cross-loading = 0.3, n=500, 5% reject proportion for the PPP = 0.006							
Major loading	0.800	0.7891	0.0493	0.0553	0.0025	0.958	1.000
Cross-loading	0.300	0.2077	0.0318	0.0545	0.0095	0.640	1.000
Factor variance	1.000	1.1116	0.1252	0.1573	0.0281	0.930	1.000
Factor correlation	0.500	0.5636	0.0368	0.0661	0.0054	0.960	1.000

Table 13: Wording and hypothesized factor loading pattern for the 15 items used to measure the big-five personality factors in the British Household Panel data ("I see myself as someone who ...")

	Agreeableness	Conscientiousness	Extraversion	Neuroticism	Openness
y1: Is sometimes rude to others (reverse-scored)	X	0	0	0	0
y2: Has a forgiving nature	X	0	0	0	0
y3: Is considerate and kind to almost everyone	X	0	0	0	0
y4: Does a thorough job	0	X	0	0	0
y5: Tends to be lazy (reverse-scored)	0	X	0	0	0
y6: Does things efficiently	0	X	0	0	0
y7: Is talkative	0	0	X	0	0
y8: Is outgoing, sociable	0	0	X	0	0
y9: Is reserved (reverse-scored)	0	0	X	0	0
y10: Worries a lot	0	0	0	X	0
y11: Gets nervous easily	0	0	0	X	0
y12: Is relaxed, handles stress well (reverse-scored)	0	0	0	X	0
y13: Is original, comes up with new ideas	0	0	0	0	X
y14: Values artistic, aesthetic experiences	0	0	0	0	X
y15: Has an active imagination	0	0	0	0	X

Table 14: ML model testing results for big-five personality factors using British Household Panel data for females ($n = 691$) and males ($n = 589$)

Model	χ^2	df	p-value	RMSEA	CFI
Females					
CFA	552	80	0.000	0.092	0.795
CFA + CUs	432	74	0.000	0.084	0.845
EFA	183	40	0.000	0.072	0.938
Males					
CFA	516	80	0.000	0.096	0.795
CFA + CUs	442	74	0.000	0.092	0.826
EFA	113	40	0.000	0.056	0.965

Table 15: ML exploratory factor analysis of the big-five personality factors using British Household Panel data

Loadings

	Females					Males				
	F1	F2	Extraver	Neurot	Open	F1	F2	Extraver	Neurot	Open
y1	**0.827***	0.000	0.014	-0.011	-0.005	**0.389***	0.010	-0.016	-0.083	-0.294*
y2	0.147*	0.215*	**0.323***	0.033	0.020	0.188	**0.447***	0.123*	-0.030	-0.011
y3	0.103*	**0.569***	0.280*	0.046	-0.095*	**0.506***	0.469*	0.026	0.042	-0.030
y4	0.018	**0.455***	-0.003	-0.025	0.270*	**0.406***	-0.011	0.119*	0.010	0.272*
y5	**0.365***	0.220*	-0.039	-0.068	0.009	**0.654***	-0.449*	-0.037	0.009	0.004
y6	-0.016	**0.852***	0.001	-0.052	0.087	**0.656***	0.077	0.020	-0.090	0.141*
y7	-0.154*	0.053	**0.541***	0.015	0.129*	0.047	0.015	**0.629***	0.045	0.123
y8	-0.041	-0.024	**0.748***	-0.049	-0.002	0.012	0.024	**0.795***	-0.051	0.032
y9	0.064	**-0.416***	0.346*	-0.116*	0.031	-0.156	-0.380*	**0.396***	-0.010	-0.049
y10	-0.045	0.061	0.063	**0.727***	0.036	0.023	0.020	-0.029	**0.698***	0.294*
y11	0.021	0.001	-0.039	**0.670***	0.013	-0.075	0.279*	-0.052	**0.519***	0.021
y12	0.022	-0.250*	-0.061	**0.547***	-0.063	-0.004	-0.311*	0.051	**0.648***	-0.109
y13	0.024	0.011	-0.036	-0.107	**0.764***	0.069	-0.007	-0.001	-0.023	**0.734***
y14	0.011	-0.088*	0.038	0.125*	**0.659***	-0.073	0.057	0.035	0.036	**0.486***
y15	-0.054	0.140*	0.087	-0.014	**0.541***	0.002	0.006	0.038	-0.146*	**0.671***

Factor correlations

	Females					Males				
	F1	F2	Extraver	Neurot	Open	F1	F2	Extraver	Neurot	Open
F1	1.000					1.000				
F2	0.151*	1.000				0.399*	1.000			
Extraver	-0.024	0.362*	1.000			0.268*	0.200*	1.000		
Neurot	-0.085	0.100*	-0.108*	1.000		-0.311*	0.065	-0.252*	1.000	
Open	-0.142*	0.229*	0.473*	-0.175*	1.000	0.344*	0.397*	0.454*	-0.180*	1.000

Table 16: Bayesian analysis using informative, small-variance priors for residual correlations using data for BHPS females and males, Method 2

Loadings

	Females					Males				
	Agreeab	Conscien	Extraver	Neurot	Open	Agreeab	Conscien	Extraver	Neurot	Open
y1	**0.772***	-0.006	-0.026	0.000	-0.012	**0.842***	-0.013	-0.011	-0.010	-0.018
y2	**0.575**	-0.014	0.021	-0.013	0.028	**0.394**	-0.006	0.024	-0.006	0.018
y3	**0.503***	0.034	0.023	0.012	-0.010	**0.479***	0.040	0.005	0.021	0.013
y4	-0.029	**0.704***	0.014	-0.003	0.024	-0.040	**0.683***	0.027	0.019	0.017
y5	0.017	**0.657***	-0.001	0.006	-0.028	0.014	**0.708***	-0.020	0.002	-0.018
y6	0.032	**0.548***	-0.015	-0.007	0.015	0.043	**0.579***	0.000	-0.036	0.007
y7	-0.008	0.014	**0.685***	0.024	0.006	-0.005	0.005	**0.748***	0.016	-0.005
y8	0.023	0.002	**0.702***	-0.017	0.003	0.024	0.011	**0.754***	-0.023	0.013
y9	-0.016	-0.021	**0.622***	-0.008	0.002	-0.025	-0.015	**0.575***	0.005	-0.005
y10	-0.003	0.025	0.022	**0.791***	0.023	0.001	0.009	0.013	**0.801***	0.044
y11	0.016	-0.007	-0.024	**0.736***	-0.008	0.012	-0.010	-0.022	**0.708***	-0.024
y12	-0.012	-0.022	-0.004	**0.695***	-0.027	-0.017	-0.006	0.003	**0.613***	-0.034
y13	0.006	0.020	0.006	-0.047	**0.780***	0.004	0.023	-0.008	0.007	**0.732***
y14	-0.008	-0.010	-0.007	0.046	**0.738***	0.004	-0.021	-0.013	0.023	**0.672***
y15	0.006	-0.006	0.011	-0.003	**0.660***	-0.011	0.001	0.031	-0.035	**0.651***

Factor correlations

	Females					Males				
	Agreeab	Conscien	Extraver	Neurot	Open	Agreeab	Conscien	Extraver	Neurot	Open
Agreeab	1.000					1.000				
Conscien	0.366*	1.000				0.319*	1.000			
Extraver	0.081	0.119	1.000			0.025	0.197	1.000		
Neurot	-0.059	-0.093	-0.163	1.000		-0.133	-0.238*	-0.160	1.000	
Open	0.041	0.201	0.321*	-0.158	1.000	0.040	0.250	0.297*	-0.091	1.000

Table 17: Rejection rates for ML CFA and Bayes CFA with non-informative priors for residual covariances

Cross-loading	Sample size	ML LRT rejection rate	Bayes PPP rejection rate
0.0	200	0.048	0.024
	500	0.070	0.024
0.1	200	0.750	0.510
	500	1.000	0.984
0.3	200	1.000	1.000
	500	1.000	1.000

Table 18: Residual correlations =0.1, Method 1

Parameter	Estimates			S.E.	M.S.E.	95%	% Sig
	Population	Average	Std. Dev.	Average		Cover	Coeff
Residual covariance = 0.1, df=30, n=200, 5% reject proportion for the PPP = 0.024							
Factor loading	0.800	0.8074	0.0690	0.0688	0.0048	0.942	1.000
Residual covariance	0.100	0.0154	0.0106	0.0159	0.0073	0.044	0.076
Factor variance	1.000	1.0257	0.1602	0.1595	0.0263	0.934	1.000
Factor correlation	0.500	0.5141	0.0599	0.0608	0.0038	0.950	1.000
Residual covariance = 0.1, df=30, n=500, 5% reject proportion for the PPP = 0.002							
Factor loading	0.800	0.8035	0.0425	0.0449	0.0018	0.958	1.000
Residual covariance	0.100	0.0399	0.0151	0.0178	0.0038	0.162	0.860
Factor variance	1.000	1.0296	0.1010	0.1016	0.0111	0.946	1.000
Factor correlation	0.500	0.5126	0.0374	0.0382	0.0016	0.940	1.000
Residual covariance = 0.1, df=14, n=200, 5% reject proportion for the PPP = 0.000							
Factor loading	0.800	0.8103	0.0694	0.0792	0.0049	0.970	1.000
Residual covariance	0.100	0.0534	0.0252	0.0376	0.0028	0.846	0.338
Factor variance	1.000	1.0139	0.1596	0.1717	0.0256	0.952	1.000
Factor correlation	0.500	0.5162	0.0605	0.0644	0.0039	0.952	1.000
Residual covariance = 0.1, df=14, n=500, 5% reject proportion for the PPP = 0.000							
Factor loading	0.800	0.8056	0.0447	0.0552	0.0020	0.976	1.000
Residual covariance	0.100	0.0665	0.0201	0.0309	0.0015	0.894	0.880
Factor variance	1.000	1.0142	0.1015	0.1148	0.0105	0.968	1.000
Factor correlation	0.500	0.5130	0.0380	0.0413	0.0016	0.952	1.000

Table 19: Residual correlations =0.3, Method 1

Parameter	Estimates			S.E.	M.S.E.	95%	% Sig
	Population	Average	Std. Dev.	Average		Cover	Coeff
Residual covariance = 0.3, df=30, n=200, 5% reject proportion for the PPP = 0.004							
Factor loading	0.800	0.8041	0.0401	0.0437	0.0016	0.970	1.000
Residual covariance	0.300	0.1803	0.0315	0.0399	0.0153	0.210	1.000
Factor variance	1.000	1.1252	0.1425	0.1464	0.0359	0.864	1.000
Factor correlation	0.500	0.5443	0.0570	0.0554	0.0052	0.870	1.000
Residual covariance = 0.3, df=30, n=500, 5% reject proportion for the PPP = 0.000							
Factor loading	0.800	0.8029	0.0264	0.0293	0.0007	0.962	1.000
Residual covariance	0.300	0.2160	0.0291	0.0349	0.0079	0.372	1.000
Factor variance	1.000	1.0855	0.0901	0.0933	0.0154	0.850	1.000
Factor correlation	0.500	0.5335	0.0357	0.0364	0.0024	0.866	1.000
Residual covariance = 0.3, df=14, n=200, 5% reject proportion for the PPP = 0.000							
Factor loading	0.800	0.8058	0.0411	0.0494	0.0017	0.974	1.000
Residual covariance	0.300	0.2307	0.0496	0.0611	0.0073	0.854	1.000
Factor variance	1.000	1.0461	0.1424	0.1554	0.0224	0.948	1.000
Factor correlation	0.500	0.5344	0.0603	0.0629	0.0048	0.900	1.000
Residual covariance = 0.3, df=14, n=500, 5% reject proportion for the PPP = 0.000							
Factor loading	0.800	0.8033	0.0278	0.0347	0.0008	0.986	1.000
Residual covariance	0.300	0.2583	0.0406	0.0483	0.0034	0.890	1.000
Factor variance	1.000	1.0199	0.0930	0.1042	0.0090	0.972	1.000
Factor correlation	0.500	0.5240	0.0386	0.0427	0.0021	0.924	1.000

Table 20: Residual correlations = 0.3, Method 2 and Method 3

Parameter	Estimates			S.E.	M.S.E.	95%	% Sig
	Population	Average	Std. Dev.	Average		Cover	Coeff
Method 2, residual covariance = 0.3, df = 30, n=200							
Factor loading	0.800	0.8041	0.0402	0.0431	0.0016	0.958	1.000
Residual covariance	0.300	0.2364	0.0394	0.0490	0.0056	0.794	1.000
Factor variance	1.000	0.9809	0.1349	0.1429	0.0185	0.946	1.000
Factor correlation	0.500	0.4817	0.0588	0.0594	0.0038	0.950	1.000
Method 2, residual covariance = 0.3, df = 30, n=500							
Factor loading	0.800	0.8029	0.0260	0.0311	0.0007	0.972	1.000
Residual covariance	0.300	0.2809	0.0348	0.0436	0.0016	0.974	1.000
Factor variance	1.000	0.9541	0.0865	0.1016	0.0096	0.952	1.000
Factor correlation	0.500	0.4760	0.0372	0.0401	0.0020	0.938	1.000
Method 3, residual covariance = 0.3, V = 0.001, n=200							
Factor loading	0.800	0.8076	0.0536	0.0587	0.0029	0.970	1.000
Residual covariance	0.300	0.1433	0.0131	0.0212	0.0247	0.000	1.000
Factor variance	1.000	1.0550	0.1482	0.1529	0.0249	0.928	1.000
Factor correlation	0.500	0.5537	0.0585	0.0578	0.0063	0.830	1.000
Method 3, residual covariance = 0.3, V = 0.001, n=500							
Factor loading	0.800	0.8047	0.0379	0.0456	0.0015	0.980	1.000
Residual covariance	0.300	0.1818	0.0119	0.0188	0.0141	0.000	1.000
Factor variance	1.000	1.0390	0.0964	0.1039	0.0108	0.956	1.000
Factor correlation	0.500	0.5499	0.0357	0.0377	0.0038	0.772	1.000

Table 21: Residual correlations = 0.3, Method 3, freeing the two residual correlations

Parameter	Estimates			S.E.	M.S.E.	95%	% Sig
	Population	Average	Std. Dev.	Average		Cover	Coeff
Residual correlations = 0.3, V = 0.001, freeing the two residual correlations, n=200							
Factor loading	0.800	0.8074	0.0463	0.0599	0.0022	0.988	1.000
Residual covariance	0.300	0.3060	0.0394	0.0429	0.0016	0.960	1.000
Factor variance	1.000	0.9959	0.1358	0.1494	0.0184	0.966	1.000
Factor correlation	0.500	0.5016	0.0600	0.0613	0.0036	0.960	1.000
Residual correlations = 0.3, V = 0.001, freeing the two residual correlations, n=500							
Factor loading	0.800	0.8074	0.0462	0.0600	0.0022	0.992	1.000
Residual covariance	0.300	0.3066	0.0395	0.0435	0.0016	0.968	1.000
Factor variance	1.000	0.9965	0.1354	0.1495	0.0183	0.968	1.000
Factor correlation	0.500	0.5017	0.0600	0.0613	0.0036	0.960	1.000

Table 22: Antisocial behavior ML EFA

Parameter	Prop	Person	Drugs
property	**0.736***	0.118*	-0.061
fight	0.267*	0.497*	-0.047
shoplift	**0.645***	-0.076	0.151*
lt50	**0.830***	-0.196*	0.003
gt50	**0.824***	-0.013	0.002
force	0.420*	0.341*	0.011
threat	0.018	**0.772***	0.163
injure	-0.012	**0.740***	0.199*
pot	-0.034	-0.013	**0.908***
drug	0.018	-0.044	**0.872***
soldpot	0.162*	0.015	**0.758***
solddrug	0.174*	0.012	**0.689***
con	**0.484***	0.191*	-0.059
auto	**0.493***	0.118*	0.081
bldg	**0.844***	-0.015	-0.003
goods	**0.732***	0.072	0.061
gambling	0.372*	0.332*	0.119

Factor correlations

	Prop	Person	Drugs
prop	1.000		
person	0.564	1.000	
drugs	0.629	0.256	1.000

Table 23: Antisocial behavior. BSEM with prior variance = 0.04. Factor solution

Parameter	Prop	Person	Drugs
property	**0.709***	0.116*	-0.059
fight	0.085	**0.611***	-0.059
shoplift	**0.679***	-0.046	0.168*
lt50	**0.862***	-0.170*	0.004
gt50	**0.828***	-0.025	-0.014
force	0.260*	**0.404***	0.007
threat	-0.047	**0.874***	0.013
injure	-0.049	**0.856***	0.026
pot	-0.038	0.009	**0.894***
drug	-0.019	-0.003	**0.896***
soldpot	0.040	0.002	**0.846***
solddrug	0.073	0.014	**0.711***
con	**0.582***	0.182*	-0.117*
auto	**0.504***	0.101*	0.039
bldg	**0.811***	-0.030	-0.004
goods	**0.763***	0.048	0.022
gambling	0.327*	0.300*	0.033

Factor correlations

Parameter	Prop	Person	Drugs
Prop	1.000		
Person	0.652*	1.000	
Drugs	0.688*	0.482*	1.000

Table 24: Antisocial behavior. BSEM with prior variance = 0.04. Effects of covariates

Parameter	Age94	Male	Black
Factors			
prop	-0.164	0.361*	-0.032
person	-0.104	0.333*	0.119*
drugs	0.093	0.133*	-0.139*
Items			
property	-0.055	0.028	**-0.051***
fight	-0.142	**0.107***	**0.052***
shoplift	-0.029	**-0.144***	0.017
lt50	0.093	-0.031	-0.048
gt50	0.078	0.013	**0.066***
force	0.002	0.025	**0.114***
threat	-0.056	-0.013	-0.066
injure	0.017	-0.053	0.007
pot	0.033	-0.028	0.029
drug	0.031	-0.051	**-0.074***
soldpot	-0.067	**0.085***	0.024
solddrug	-0.012	0.015	0.061
con	-0.019	**-0.140***	**0.075***
auto	-0.042	-0.048	-0.029
bldg	0.027	**0.082***	-0.040
goods	0.049	0.022	-0.018
gambling	0.089	**0.077***	0.033

Table 25: Bayesian analysis with informative, small-variance priors for direct effects

Parameter	Estimates			S.E.	M.S.E.	95%	% Sig
	Population	Average	Std. Dev.	Average		Cover	Coeff
Direct effect = 0.4, n=500, 5% reject proportion for the PPP = 0.000							
Major loading	1.000	1.0981	0.2030	0.2009	0.0508	0.930	1.000
Factor slope on X	0.500	0.5277	0.1265	0.1489	0.0167	0.974	0.994
Direct effect of Y on X	0.400	0.2456	0.0850	0.1328	0.0310	0.884	0.422
Factor residual variance	1.000	0.9065	0.2539	0.2371	0.0731	0.892	1.000
Factor residual correlation	0.500	0.5066	0.0530	0.0557	0.0028	0.952	1.000
Direct effect = 0.4, n=1000, 5% reject proportion for the PPP = 0.000							
Major loading	1.000	1.0462	0.1360	0.1357	0.0206	0.950	1.000
Factor slope on X	0.500	0.5602	0.1000	0.1242	0.0136	0.958	1.000
Direct effect of Y on X	0.400	0.2778	0.0722	0.1161	0.0201	0.930	0.700
Factor residual variance	1.000	0.9506	0.1792	0.1759	0.0345	0.910	1.000
Factor residual correlation	0.500	0.5018	0.0393	0.0394	0.0015	0.946	1.000
Direct effect = 0.4, n=2000, 5% reject proportion for the PPP = 0.000							
Major loading	1.000	1.0220	0.0955	0.0937	0.0096	0.942	1.000
Factor slope on X	0.500	0.5676	0.0730	0.1061	0.0099	0.972	1.000
Direct effect of Y on X	0.400	0.2976	0.0581	0.1032	0.0138	0.958	0.932
Factor residual variance	1.000	0.9815	0.1350	0.1282	0.0185	0.926	1.000
Factor residual correlation	0.500	0.5012	0.0287	0.0279	0.0008	0.940	1.000