

# IRT in Mplus

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In this note we describe several of the IRT modeling features implemented in Mplus, namely the the item characteristic curves, the item information curves, the total information curve, item difficulty parameter and item discrimination parameter.

## 1 ICC curves

### 1.1 Logit Link, ML/MLR/MLF Estimators

Let  $U_i$  be a categorical indicator for a latent factor  $f$  in the presence of a categorical latent class variable  $C$ . The item characteristic curves (ICC) for the item  $U_i$ , given that  $C = k$  are computed as follows using the logistic model. If the category  $j$  is the first category

$$P_{ijk}(f) = P(U_i = j|f, C = k) = \frac{1}{1 + \text{Exp}(-\tau_{ijk} + \lambda_{ik}f)}. \quad (1)$$

If the category  $j$  is the last category

$$P_{ijk}(f) = P(U_i = j|f, C = k) = 1 - \frac{1}{1 + \text{Exp}(-\tau_{ij-1k} + \lambda_{ik}f)}. \quad (2)$$

If the category  $j$  is a middle category

$$P_{ijk}(f) = P(U_i = j|f, C = k) = \frac{1}{1 + \text{Exp}(-\tau_{ijk} + \lambda_{ik}f)} - \frac{1}{1 + \text{Exp}(-\tau_{ij-1k} + \lambda_{ik}f)}. \quad (3)$$

In the presence of other covariates/other latent variables  $X$  the formulas are modified as follows. If the category  $j$  is the first category

$$P_{ijk}(f) = P(U_i = j|f, C = k, X = x) = \frac{1}{1 + \text{Exp}(-\tau_{ijk} + \lambda_{ik}f + \beta_{ik}x)}. \quad (4)$$

If the category  $j$  is the last category

$$P_{ijk}(f) = P(U_i = j|f, C = k, X = x) = 1 - \frac{1}{1 + \text{Exp}(-\tau_{ij-1k} + \lambda_{ik}f + \beta_{ik}x)}. \quad (5)$$

If the category  $j$  is a middle category

$$P_{ijk}(f) = P(U_i = j|f, C = k, X = x) = \frac{1}{1 + \text{Exp}(-\tau_{ijk} + \lambda_{ik}f + \beta_{ik}x)} - \frac{1}{1 + \text{Exp}(-\tau_{ij-1k} + \lambda_{ik}f + \beta_{ik}x)}. \quad (6)$$

## 1.2 Probit Link, ML/MLR/MLF Estimators

Let  $\Phi$  be the standard normal cumulative distribution function. The ICC curves are given as follows. If the category  $j$  is the first category

$$P_{ijk}(f) = P(U_i = j|f, C = k, X = x) = \Phi(\tau_{ijk} - \lambda_{ik}f - \beta_{ik}x). \quad (7)$$

If the category  $j$  is the last category

$$P_{ijk}(f) = P(U_i = j|f, C = k, X = x) = 1 - \Phi(\tau_{ij-1k} - \lambda_{ik}f - \beta_{ik}x). \quad (8)$$

If the category  $j$  is a middle category

$$P_{ijk}(f) = P(U_i = j|f, C = k, X = x) = \Phi(\tau_{ijk} - \lambda_{ik}f - \beta_{ik}x) - \Phi(\tau_{ij-1k} - \lambda_{ik}f - \beta_{ik}x). \quad (9)$$

### 1.3 Probit Link, WLS/WLSM/WLSMV/ULS Estimators, Theta Parametrization

In this situation the model does not include latent categorical variable  $C$  however multiple group models are included. Let  $G$  denote the group variable. With the Theta parametrization the residual parameter  $\theta_{ik}$  is an actual parameter in the model. For basic models this parameter is fixed to 1 since it will not be identified without model restrictions, however for multiple group and growth models the parameter could be identified. If these parameters are not printed in the results section that means that they are fixed to 1. The ICC curves are given as follows. If the category  $j$  is the first category

$$P_{ijk}(f) = P(U_i = j|f, G = k, X = x) = \Phi\left(\frac{\tau_{ijk} - \lambda_{ik}f - \beta_{ik}x}{\sqrt{\theta_{ik}}}\right). \quad (10)$$

If the category  $j$  is the last category

$$P_{ijk}(f) = P(U_i = j|f, G = k, X = x) = 1 - \Phi\left(\frac{\tau_{ij-1k} - \lambda_{ik}f - \beta_{ik}x}{\sqrt{\theta_{ik}}}\right). \quad (11)$$

If the category  $j$  is a middle category

$$P_{ijk}(f) = P(U_i = j|f, G = k, X = x) = \Phi\left(\frac{\tau_{ijk} - \lambda_{ik}f - \beta_{ik}x}{\sqrt{\theta_{ik}}}\right) - \Phi\left(\frac{\tau_{ij-1k} - \lambda_{ik}f - \beta_{ik}x}{\sqrt{\theta_{ik}}}\right). \quad (12)$$

### 1.4 Probit Link, WLS/WLSM/WLSMV/ULS Estimators, Delta Parametrization

With the Delta parametrization the  $\theta_{ik}$  are not actual parameters but are dependent parameters that are obtained from the following equation

$$\theta_{ik} = \Delta_{ik}^{-2} - Var(\lambda_{ik}f)$$

where  $\Delta_{ik}$  are actual parameters that can be either free or fixed. Again the  $\Delta_{ik}$  are typically not identifiable and are fixed to 1, however in growth and multiple group models the parameter can be free and identified. When the  $\Delta_{ik}$  parameters are not present in the results, they are fixed to 1. The  $\theta_{ik}$  parameters are always reported in the results section and are typically smaller than 1. For example when the  $\Delta_{ik}$  parameters are fixed to 1 the  $\theta_{ik}$  are smaller than 1. The ICC curves are given as in the previous section.

## 2 IIC curves

The item information curves (IIC) for a categorical indicator  $U_i$  and a latent factor  $f$  in class  $C = k$  is computed as follows. Let  $y_{ijk}$  be the sample frequency for category  $j$  in class  $k$ . In computing these frequencies we use the posterior class distribution for each observation to allocate each observation partially within each of the possible classes. When there is only one class these sample frequencies are the observed sample frequencies. Define the cumulative sample frequencies as  $x_{ijk} = \sum_{r=1}^j y_{irk}$ . If the total number of categories for  $U_i$  are  $l$  then  $x_{i0k} = 0$  and  $x_{ilk} = 1$ . Also define for  $1 \leq j \leq l-1$

$$Q_{ijk} = \sum_{r=1}^j P_{irk}. \quad (13)$$

The IIC is the likelihood information function (second derivative of the log-likelihood with respect to  $f$ ). For the ML/MLF/MLR estimators with the logit link functions the IIC curve is given by

$$I_i(f) = \lambda_{ik}^2 \sum_{j=1}^{l-1} (x_{ij+1k} - x_{ij-1k}) Q_{ijk} (1 - Q_{ijk}). \quad (14)$$

For the ML/MLF/MLR estimators with the probit link functions the IIC curve is given by

$$I_i(f) = 3.29 \cdot \lambda_{ik}^2 \sum_{j=1}^{l-1} (x_{ij+1k} - x_{ij-1k}) Q_{ijk} (1 - Q_{ijk}). \quad (15)$$

For the WLS/WLSM/WLSMV/ULS estimators with the probit link functions and either theta or delta parametrization the IIC curve is given by

$$I_i(f) = 3.29 \cdot \frac{\lambda_{ik}^2}{\theta_{ik}} \sum_{j=1}^{l-1} (x_{ij+1k} - x_{ij-1k}) Q_{ijk} (1 - Q_{ijk}). \quad (16)$$

The total information function is obtained by adding all item information functions

$$I(f) = \sum_i I_i(f). \quad (17)$$

The above formulas are obtained using the logit link function and are exact in this case. They are simply an approximation for the probit link function. The

constant 3.29 used with the probit link function is simply the  $\pi^2/3$  constant which is needed to adjust the scale of the loadings.

The information function  $I(f)$  can be used to calculate standard errors for the factor score estimates

$$SE(f) = \frac{1}{\sqrt{I(f)}}.$$

This is because  $I(f)$  is the expected information function. To obtain the standard errors for the factor score in the presence of missing data the total information function in the above formula is replaced by the sum of the IIC for the indicators that are present for that observation. These factor score standard errors can also be obtained in Mplus directly using the `SAVEDATA` command and the `ML/MLF/MLR` estimators with numerical integration from the estimated posterior distribution of the factor. The two methods differ to some extent but generally yield approximately the same results. A third method for computing the factor score standard errors is with the Bayes estimator where the posterior distribution for each factor can also be estimated.

### 3 IRT Parameterization

For binary items with a single factor we provide the parameter estimates also in the traditional IRT scale. Let the factor mean be  $\alpha$  and the factor variance be  $\psi$ . Thus  $f = \alpha + \sqrt{\psi}\theta$  where  $\theta$  is the IRT standard normal latent variable with mean 0 and standard deviation 1. For the `ML/MLF/MLR` estimators with the logit link function

$$P(U_i = 1|f) = \frac{1}{1 + \text{Exp}(\tau_{ik} - \lambda_{ik}f)} = \frac{1}{1 + \text{Exp}(-Da_{ik}(\theta - b_{ik}))} \quad (18)$$

where  $D = 1.7$  is a constant that gives the IRT estimates close to the probit scale,  $a_{ik}$  is the item discrimination parameter and  $b_{ik}$  is the item difficulty parameter. These parameters are computed as follows

$$a_{ik} = \frac{\lambda_{ik}\sqrt{\psi}}{D} \quad (19)$$

$$b_{ik} = \frac{\tau_{ik} - \lambda_{ik}\alpha}{\lambda_{ik}\sqrt{\psi}}. \quad (20)$$

For the other estimations, links and parametrization the IRT parametrization is obtained by the same approach. The resulting formulas for  $b_{ik}$  is the same as (20), while the parameter  $a_{ik}$  is obtained as follows. For the ML/MLF/MLR estimators with the probit link function

$$a_{ik} = \lambda_{ik} \sqrt{\psi}. \quad (21)$$

For the WLS/WLSM/WLSMV/ULS estimators with the probit link functions and theta parametrization

$$a_{ik} = \frac{\lambda_{ik} \sqrt{\psi}}{\sqrt{\theta_{ik}}} \quad (22)$$

For the WLS/WLSM/WLSMV/ULS estimators with the probit link functions and delta parametrization

$$a_{ik} = \frac{1}{\sqrt{\Delta_{ik}^{-2} \lambda_{ik}^{-2} \psi^{-1} - 1}} \quad (23)$$

The standard errors of these parameters are computed by the delta method.

## 4 References

The following references can be used for additional information on the IRT model.

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