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CONTRIBUTIONS TO FACTOR ANALYSIS OF DICHOTOMOUS VARIABLES

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A new method is proposed for the factor analysis of dichotomous variables. Similar to the method of Christoffersson this uses information from the first and second order proportions to fit a multiple factor model. Through a transformation into a new set of sample characteristics, the estimation is considerably simplified. A generalized least-squares estimator is proposed, which asymptotically is as efficient as the corresponding estimator of Christoffersson, but which demands less computing time.

Key words: multiple factor model, first and second order proportions, generalized least-squares, tetrachoric correlations.

Maximum likelihood estimation of the factor analysis model for dichotomous variables was treated by Bock & Lieberman [1970] for the special case of one factor. Despite the limitation to the one factor case, the estimation method is computationally extremely heavy, and the practical maximum limit of variables that can be handled is around 10–12. Recently, Christoffersson [1975] generalized the model to multiple factors and presented a simpler approach to estimation. Christoffersson [1975] proposed the use of response information only regarding the first order marginal distributions. In this way, many more variables can be handled in the analysis, and thus a more feasible analysis procedure was made available.

In this paper we present still another approach to estimation, which uses the same amount of information as Christoffersson [1975] but leads to yet simpler computations. A generalized least squares (GLS) estimator will be proposed, which will be shown to be asymptotically as efficient as the GLS estimator of Christoffersson [1975]. This new estimator has in fact a close relation to the traditional, heuristic, method of fitting the factor model, namely using the normal deviates corresponding to the sample proportions, and using the sample tetrachoric correlation coefficients. For both GLS estimators the practical limit for the number of variables is around 20–25, the limitation being mainly due to the largeness of the weight matrix. The use of the new GLS estimator gives a large reduction in computing time. While this gain may not be of great practical value for psychological tests which often have considerably more than 20–25 items, it is valuable for social- psychological applications such as attitude studies, where the number of items is relatively small.

The Model

Denote by $\mathbf{u}$ the p-dimensional vector of dichotomous variables, each of which has two possible response alternatives, 0 and 1. Underlying each observed variable u_i ($i = 1, 2, \cdots, p$) it is assumed a latent variable, a so called response strength, ξ_i^* ($i = 1, 2, \cdots, p$), so

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that

$$u_i = \begin{cases} 1, & \text{if } \xi_i^* \geq \tau_i \\ 0, & \text{if } \xi_i^* < \tau_i , \end{cases} \tag{1}$$

$i = 1, 2, \cdots, p$, where the parameter τ_i is interpreted as a threshold value for ξ_i^*. We also assume that

$$\boldsymbol{\xi}^* = \boldsymbol{\Lambda\xi} + \boldsymbol{\varepsilon} , \tag{2}$$

where $\boldsymbol{\Lambda}$ is a $p \times k$ matrix of factor loadings, $\boldsymbol{\xi}$ is the k-dimensional latent variable vector of the factors, and $\boldsymbol{\varepsilon}$ is a p-dimensional vector of residuals that is uncorrelated with $\boldsymbol{\xi}$ and has zero expectation. It is further assumed that $\boldsymbol{\xi}^*$ is multivariate normal with zero expectation. We obtain the covariance matrix

$$\begin{aligned} V(\boldsymbol{\xi}^*) &= \boldsymbol{\Sigma} \\ &= \boldsymbol{\Lambda\Phi\Lambda}' + \boldsymbol{\Psi} , \end{aligned} \tag{3}$$

where $\boldsymbol{\Phi}$ is the covariance matrix of the factors, and $\boldsymbol{\Psi}$ is the covariance matrix of ε, assumed to be diagonal. The elements of $\boldsymbol{\Psi}$ are not free parameters, but

$$\boldsymbol{\Psi} = I - \operatorname{diag}(\boldsymbol{\Lambda\Phi\Lambda}') , \tag{4}$$

yielding diag $(\boldsymbol{\Sigma}) = \mathbf{I}$. This restriction is necessary since there is no possibility to identify the diagonal elements of $\boldsymbol{\Sigma}$, only observing dichotomous variables. The model thus contains three parameter arrays: $\boldsymbol{\tau}$, $\boldsymbol{\Lambda}$, and $\boldsymbol{\Phi}$.

Let σ_{ij} be the i, j^{th} element of $\boldsymbol{\Sigma}$ given by (3). From the model we deduce that

$$P(u_i = 1) = \int_{\tau_i}^{\infty} \phi(z)\, dz , \tag{5}$$

$$P(u_i = 1, u_j = 1) = \int_{\tau_i}^{\infty}\int_{\tau_j}^{\infty} \phi(z_1, z_2; \sigma_{ij})\, dz_1\, dz_2 , \tag{6}$$

$i, j = 1, 2 \cdots, p(i \neq j)$, where $\phi(z_1, z_2; \rho)$ denotes the density of a standard bivariate normal distribution with correlation ρ. Christoffersson [1975] used the corresponding sample proportions p_i and p_{ij} to fit the $m = p(p + 1)/2$ relations

$$p_i = P(u_i = 1) + \epsilon_i, \qquad i = 1, 2, \cdots, p \tag{7}$$

$$p_{ij} = P(u_i = 1, u_j = 1) + \epsilon_{ij}, \qquad i = 2, 3, \cdots, p \qquad j = 1, 2, \cdots, i - 1 \tag{8}$$

where the ϵ's may be stacked in the m-dimensional vector $\boldsymbol{\varepsilon}$ with expectation zero, and covariance matrix $\boldsymbol{\Sigma}_\epsilon$, given by Appendix 2 of Christoffersson [1975]. Denote the sample size by N. Using a consistent estimate of $N \cdot \boldsymbol{\Sigma}_\epsilon$ to form the weight matrix, his GLS estimator employs all m relations of type (7) and (8) to estimate the parameters (see Christoffersson, 1975, page 8). This estimator is shown to be consistent and asymptotically efficient among those estimators that use the same amount of information. It demands however the repeated calculation of the single and double integrals of (5) and (6), and their derivatives. This task is a large part in each iterative calculation, and it is time consuming for larger models.

We will now present a solution to the estimation problem, which avoids the iterated computation of these integrals. We define the one-to-one transformation from $x_1, x_2, \cdots, x_p, x_{21}, x_{31}, x_{32}, \cdots, x_{p,p-1}$ to $y_1, y_2, \cdots, y_p, y_{21}, y_{31}, y_{32}, \cdots, y_{p,p-1}$:

$$y_i = \int_{x_i}^{\infty} \phi(z)\, dz , \tag{9}$$

$$y_{ij} = \int_{x_i}^{\infty}\int_{x_j}^{\infty} \phi(z_1, z_2; x_{ij})\, dz_1\, dz_2 \cdot \tag{10}$$

Let this transformation be denoted $\mathbf{y} = \mathbf{f}(\mathbf{x})$, where $\mathbf{y}$ and $\mathbf{x}$ are of order $m \times 1$. The inverse transformation is $\mathbf{x} = \mathbf{f}^{-1}(\mathbf{y})$.

Returning to the relations of (7) and (8), we may arrange the observed proportions in the m-dimensional vector $\mathbf{p}$, and arrange the right-hand-side terms accordingly. Using the transformation $\mathbf{f}$, we may write

$$\mathbf{p} = \mathbf{f}(\boldsymbol{\theta}) + \boldsymbol{\varepsilon}\,, \tag{11}$$

where $\boldsymbol{\theta} = (\boldsymbol{\theta}_1', \boldsymbol{\theta}_2')$ is an m-dimensional vector with $\boldsymbol{\theta}_1 = \boldsymbol{\tau}$, and $\boldsymbol{\theta}_2$ denoting the vector of elements below the diagonal of $\boldsymbol{\Sigma} = \boldsymbol{\Lambda\Phi\Lambda}' + \boldsymbol{\Psi}$ from (3). We recognize $\boldsymbol{\theta}_1$ as the vector of population thresholds, and $\boldsymbol{\theta}_2$ as the vector of population tetrachoric correlations.

Instead of estimating the parameters of $\boldsymbol{\tau}$, $\boldsymbol{\Lambda}$, and $\boldsymbol{\Phi}$ by direct fitting of (11), we propose to estimate simply by fitting $\boldsymbol{\theta}$ to the corresponding sample vector $\mathbf{t}$, say, where

$$\mathbf{t} = \mathbf{f}^{-1}(\mathbf{p})\,, \tag{12}$$

with the first p elements of $\mathbf{t}$ containing the sample thresholds and the last $p(p - 1)/2$ elements containing the sample tetrachoric correlations. For the computation of $\mathbf{t}$ given $\mathbf{p}$ we will use an efficient algorithm developed by Kirk [1973]. Applying this, we employ eight-point Gaussian quadrature, and Newton-Raphson iterations with the convergence criteria used by Kirk [1973, p. 263].

Applying the f-transformation to (11) and making a Taylor expansion around $\mathbf{f}$, we find the "linearized" expression

$$\mathbf{t} = \mathbf{f}^{-1}[\mathbf{f}(\boldsymbol{\theta}) + \boldsymbol{\varepsilon}] = \boldsymbol{\theta} + \boldsymbol{\Gamma\varepsilon} + \mathbf{r}\,, \tag{13}$$

where $\boldsymbol{\Gamma}$ is the $m \times m$ matrix of first order derivatives $\partial \mathbf{f}^{-1}(\mathbf{f})/\partial \mathbf{f}'$, and $\mathbf{r}$ is a rest term vector containing higher-order terms. Hereby the complex non-linear relations of (11) are reduced to

$$\mathbf{t} = \boldsymbol{\theta} + \boldsymbol{\delta}\,, \tag{14}$$

where $\boldsymbol{\theta}$ thus contains the parameters of our model, and $\boldsymbol{\delta} = \boldsymbol{\Gamma\varepsilon} + \mathbf{r}$ is an m-dimensional residual vector. Since the elements of $\mathbf{t}$ are functions of sample moments, it follows from Cramér [1946, chap. 27–28] that $\boldsymbol{\delta} = \mathbf{t} - \boldsymbol{\theta}$ is asymptotically normal with mean vector zero and covariance matrix $\boldsymbol{\Gamma\Sigma_\varepsilon\Gamma}'$.

The GLS Estimator

A consistent estimator $N\cdot\mathbf{S}$ of $N\cdot\boldsymbol{\Sigma}_\varepsilon$ is available (see Christoffersson, 1975, Appendix 2). The estimator $\mathbf{S}$ uses information also from third- and fourth-order marginal proportions. A consistent estimator $\mathbf{G}$ of $\boldsymbol{\Gamma}$ is obtained replacing $\boldsymbol{\theta}$ by $\mathbf{t}$. The derivatives contained in $\mathbf{G}$ are as follows. We first note that

$$\begin{aligned} \mathbf{G} &= \frac{\partial \mathbf{t}}{\partial \mathbf{p}'} \\ &= \left[\frac{\partial \mathbf{p}}{\partial \mathbf{t}'}\right]^{-1}. \end{aligned} \tag{15}$$

Partition G^{-1} in accordance with the partitioning of $\boldsymbol{\theta} = (\boldsymbol{\theta}_1', \boldsymbol{\theta}_2')$,

$$\begin{aligned} \frac{\partial \mathbf{p}}{\partial \mathbf{t}'} &= \mathbf{G}^{-1} \\ &= \begin{bmatrix} \mathbf{G}^{11} & \mathbf{G}^{12} \\ \mathbf{G}^{21} & \mathbf{G}^{22} \end{bmatrix}, \end{aligned}$$

where the dimensions are $p \times p$ for $\mathbf{G}^{11}$, $p(p - 1)/2 \times p$ for $\mathbf{G}^{21}$, and $p(p - 1)/2 \times p(p - 1)/2$ for $\mathbf{G}^{22}$.

By (9) we find that $\mathbf{G}^{11}$ is diagonal with i, i^{th} element

(16) $$\frac{\partial p_i}{\partial t_i} = -\phi(t_i)\,,$$

$i = 1, 2, \cdots, p$ and that the elements of $\mathbf{G}^{12}$ are all zero. By (10) we obtain the elements of $\mathbf{G}^{21}$ as

(17) $$\frac{\partial p_{ij}}{\partial t_r} = \begin{cases} -\phi(\tau_i)\displaystyle\int_{v_{ij}}^{\infty} \phi(z)\,dz, & \text{if} \quad r = i \\ -\phi(\tau_j)\displaystyle\int_{w_{ij}}^{\infty} \phi(z)\,dz, & \text{if} \quad r = j \\ 0, \quad \text{otherwise} \end{cases}$$

where $v_{ij} = (t_j - t_{ij}t_i)(1 - t_{ij}^2)^{-1/2}$, $w_{ij} = (t_i - t_{ij}t_j)(1 - t_{ij}^2)^{-1/2}$, $i = 2, 3, \cdots, p, j = 1, 2, \cdots, i - 1$, and $r = 1, 2, \cdots, p$. $\mathbf{G}^{22}$ is diagonal with elements (see e.g. Kirk, 1973, pp. 259–261)

(18) $$\frac{\partial p_{ij}}{\partial t_{ij}} = (2\pi)^{-1}\cdot(1 - t_{ij})^{-1/2}\cdot e^{-1/2[(t_i^2 - 2t_it_jt_{ij} + t_j^2)\cdot(1 - t_{ij}^2)^{-1}]}$$

where the i, j-indices run as in (17).

Consider the (modified) generalized least squares fitting function

(19) $$F = \tfrac{1}{2}\,(\mathbf{t} - \boldsymbol{\theta})'\mathbf{W}^{-1}(\mathbf{t} - \boldsymbol{\theta})\,,$$

where $\mathbf{W}^{-1} = \mathbf{G}^{-1\prime}\mathbf{S}^{-1}\mathbf{G}^{-1}$. Since $N\cdot\mathbf{W}$ is a consistent estimator of $N\cdot\boldsymbol{\Gamma}\boldsymbol{\Sigma}_\epsilon\boldsymbol{\Gamma}'$, minimizing F with respect to the parameters of $\boldsymbol{\tau}$, $\boldsymbol{\Lambda}$, and $\boldsymbol{\Phi}$ yields a GLS estimator which is consistent and asymptotically efficient among estimators using the same information [c.f. Christoffersson, 1975]. We should note that the information contained in the sample thresholds and tetrachoric correlations of $\mathbf{t}$ is equivalent to the information in $\mathbf{p}$, since they have a one-to-one relationship by the transformation $\mathbf{f}$. Thus, the proposed GLS estimator is as efficient as the GLS estimator of Christoffersson [1975].

Our approach of linearization gives an additional simplification, which is also important in reducing computing time. We may also partition $\mathbf{t}$ and $\mathbf{W}^{-1}$ in accordance with $\boldsymbol{\theta}' = (\boldsymbol{\theta}_1', \boldsymbol{\theta}_2')$, putting

$$\mathbf{W}^{-1} = \begin{bmatrix} \mathbf{W}^{11} & \mathbf{W}^{12} \\ \mathbf{W}^{21} & \mathbf{W}^{22} \end{bmatrix}.$$

From the first-order condition $\partial F/\partial\boldsymbol{\tau} = \partial F/\partial\boldsymbol{\theta}_1 = \mathbf{0}$, we obtain

(20) $$\boldsymbol{\theta}_1 = \mathbf{t}_1 + [\mathbf{W}^{11}]^{-1}\,\mathbf{W}^{12}(\mathbf{t}_2 - \boldsymbol{\theta}_2)\,\cdot$$

Since there is a one-to-one relation between $\boldsymbol{\theta}_1$ and the (unconstrained) parameter vector $\boldsymbol{\tau}$, and since $\boldsymbol{\tau}$ does not appear in $\boldsymbol{\theta}_2$, we can insert (20) in F and perform the iterative minimization of F with respect to the parameters of $\boldsymbol{\Lambda}$ and $\boldsymbol{\Phi}$ only.

The minimization of F is carried out using the Fletcher-Powell method (see Fletcher & Powell, 1963) as modified by Gruvaeus & Jöreskog [Note 1]. The necessary calculations are done by a computer program FADIC, developed by Muthén & Dahlqvist [Note 2].

With $\boldsymbol{\delta} = \mathbf{t} - \boldsymbol{\theta}$ we have

(21) $$F = \tfrac{1}{2}\,\boldsymbol{\delta}'\mathbf{W}^{-1}\boldsymbol{\delta}\,\cdot$$

Consider F as a function of the elements of $\boldsymbol{\Lambda}$ and $\boldsymbol{\Phi}$ denoted by the vector $\boldsymbol{\pi}^*$, say. The elements of $\boldsymbol{\pi}^*$ may be of three kinds: free parameters, parameters fixed to a certain value, and parameters constrained to be equal to other parameters. Denote by $\boldsymbol{\pi}$ the vector of the distinct, nonfixed elements of π^*. Let $\partial\pi^*/\partial\pi' = \mathbf{K}$, a matrix of zeroes and ones. We then have

$$\frac{\partial F}{\partial \pi'} = \frac{\partial F}{\partial \boldsymbol{\theta}'} \cdot \frac{\partial \boldsymbol{\theta}}{\partial \boldsymbol{\theta}_2'} \cdot \frac{\partial \boldsymbol{\theta}_2}{\partial \pi^{*\prime}} \cdot \mathbf{K}\,, \tag{22}$$

where

$$\frac{\partial F}{\partial \boldsymbol{\theta}'} \cdot \frac{\partial \boldsymbol{\theta}}{\partial \boldsymbol{\theta}_2'} = -\boldsymbol{\delta}'\,\mathbf{W}^{-1}\begin{bmatrix} -[\mathbf{W}_{11}]^{-1} & \mathbf{W}^{12} \\ & \mathbf{I} \end{bmatrix}. \tag{23}$$

The elements of $\partial\boldsymbol{\theta}_2/\partial\pi^{*\prime}$ are obtained from

$$\frac{\partial \theta_{ij}}{\partial \lambda_{rs}} = \begin{cases} \sum_{t=1}^{k} \varphi_{st}\cdot\lambda_{it}, & \text{if} \quad r = j \\ \sum_{t=1}^{k} \varphi_{st}\cdot\lambda_{jt}, & \text{if} \quad r = i \\ 0, & \text{otherwise}\,, \end{cases} \tag{24}$$

$$\frac{\partial \theta_{ij}}{\partial \varphi_{rs}} = \begin{cases} \lambda_{ir}\cdot\lambda_{js} + \lambda_{is}\cdot\lambda_{jr}, & \text{if} \quad r \neq s \\ \lambda_{ir}\cdot\lambda_{jr}, & \text{if} \quad r = s, \end{cases} \tag{25}$$

where λ_{rs}, φ_{rs} denote the r, s^{th} elements of Λ and $\boldsymbol{\Phi}$ respectively, and $i = 2, 3, \cdots, p, j = 1, 2, \cdots, i - 1$.

Let $\boldsymbol{\gamma}' = (\boldsymbol{\tau}', \pi')$, $\mathbf{B} = \partial\boldsymbol{\theta}_2/\partial\pi'$, $\mathbf{C} = \partial\boldsymbol{\theta}/\partial\boldsymbol{\gamma}'$, and consider the asymptotic covariance matrix of the GLS estimator. Similar to Christoffersson [1975, p. 27] we find that the asymptotically normal estimator has a limiting covariance matrix that may be estimated by $[\mathbf{C}'\,\mathbf{W}^{-1}\,\mathbf{C}]^{-1}$, or

$$\begin{bmatrix} \mathbf{W}^{11} & \mathbf{W}^{12}\mathbf{B} \\ \mathbf{B}'\mathbf{W}^{21} & \mathbf{B}'\mathbf{W}^{22}\mathbf{B} \end{bmatrix}^{-1}, \tag{26}$$

evaluated at the minimum of $F(F_{\min})$. For a test of the model we use $2F_{\min}$. Directly analogous to Christoffersson [1975, pp. 27–28] it may be shown that $2F_{\min}$ is asymptotically chi-square distributed with degrees of freedom equal to m minus the number of elements in $\boldsymbol{\gamma}$.

The traditional estimation method estimates τ_i as the normal deviate corresponding to p_i $(i = 1, 2, \cdots, p)$ and separately estimates $\boldsymbol{\Lambda}$ and $\boldsymbol{\Phi}$ using the tetrachoric correlations. When the tetrachoric correlations are fitted by unweighted least-squares, this estimator, which we will call ULS, can be viewed as a special case of our method with $\mathbf{W} = \mathbf{I}$ in (19). ULS fits the same population vector $\boldsymbol{\theta}$ as GLS, but the weight matrix of GLS ensures that the standard errors of the estimates are as small as possible. The difference between these estimators is made explicit for the threshold parameters by (20), the difference being $[\mathbf{W}^{11}]^{-1}\,\mathbf{W}^{12}(\mathbf{t}_2 - \hat{\boldsymbol{\theta}}_2)$, where $\hat{\boldsymbol{\theta}}_2$ contains the tetrachoric correlations estimated by GLS. For "unrestricted models", i.e. when we only apply the k^2 restrictions on $\boldsymbol{\Lambda}$ and $\boldsymbol{\Phi}$ that are necessary for identification, ULS can be very efficiently computed. In fitting the correlations, the minimization can be carried out with respect to only p variables (see e.g. Jöreskog, 1977). It is the author's experience that the ULS-solution often approximates the GLS-solution reasonably well. For models involving a large number of items, ULS therefore seems to be a good choice as a substitute method for GLS.

We may also note that the proposed method can be modified to allow for guessing in the sense of Lord & Novick [1968, p. 404], assuming known guessing probabilities. It is only necessary to modify the calculations of the sample vector $\mathbf{t}$ and the weight matrix in a straightforward way.

Examples

We will first study two sets of data previously analyzed by Bock & Lieberman [1970] and by Christoffersson [1975]. These consist of items selected from Sections 6 and 7 of the Law School Admission Test. The number of cases is 1000. Here, we can compare the GLS estimator with the maximum likelihood (ML) estimator of Bock & Lieberman [1970], and with the GLS estimator of Christoffersson [1975], called GLS (old).

For each of the two sections, a one-factor model was fitted. For Section 6, GLS gave a chi-square value of 5.085 with 5 degrees of freedom and probability level (p-level) .406. For Section 7 the chi-square was 10.697 with 5 degrees of freedom and probability level .058. This agrees with the p-levels of the ML estimator, $0.40 < p < 0.50$ for Section 6, and $.05 < p < .10$ for Section 7 (this agreement was also obtained with GLS (old)). We should

TABLE 1

Item parameters estimated by four methods using the data of Bock & Lieberman [1970].*

	Thresholds (τ)				Loadings (Λ)			
Item	ML	GLS(old)	GLS	ULS	ML	GLS(old)	GLS	ULS
Section 6								
1	-1.4329 (.0560)	-1.448 (.059)	-1.440 (.058)	-1.433	.3856 (.1077)	.380 (.112)	.385 (.110)	.373
2	- .5505 (.0417)	- .549 (.042)	- .547 (.042)	- .551	.3976 (.0827)	.412 (.084)	.414 (.082)	.409
3	- .1332 (.0397)	- .138 (.040)	- .137 (.040)	- .133	.4732 (.0867)	.457 (.085)	.454 (.082)	.482
4	- .7159 (.0429)	- .718 (.044)	- .714 (.043)	- .716	.3749 (.0834)	.381 (.084)	.388 (.083)	.373
5	-1.1264 (.0480)	-1.139 (.050)	-1.129 (.050)	-1.126	.3377 (.0920)	.344 (.094)	.355 (.092)	.326
Section 7								
1	- .9462 (.0468)	- .964 (.047)	- .948 (.047)	- .946	.4887 (.0643)	.492 (.064)	.507 (.063)	.519
2	- .4073 (.0409)	- .409 (.041)	- .405 (.041)	- .407	.5436 (.0617)	.557 (.062)	.555 (.060)	.513
3	- .7451 (.0441)	- .761 (.044)	- .755 (.044)	- .746	.7022 (.0679)	.692 (.064)	.692 (.062)	.678
4	- .2683 (.0404)	- .272 (.046)	- .270 (.040)	- .269	.4196 (.0573)	.427 (.056)	.433 (.055)	.433
5	-1.0069 (.0479)	-1.020 (.048)	-1.007 (.048)	-1.007	.3805 (.0670)	.369 (.068)	.382 (.065)	.395

* Standard errors are given in parentheses.

TABLE 2
Items from the personality inventory of Rotter [1966].

		Items*
1	2.*a.*	Many of the unhappy things in people's lives are partly due to bad luck.
	b.	People's misfortunes results from the mistakes they make.
2	3.a.	One of the major reasons why we have wars is because people don't take enough interest in politics.
	b.	There will always be wars, no matter how hard people try to prevent them.
3	4.a.	In the long run people get the respect they deserve in this world.
	b.	Unfortunately, an individual's worth often passes unrecognized no matter how hard he tries.
4	5.a.	The idea that teachers are unfair to students is nonsense.
	b.	Most students don't realize the extent to which their grades are influenced by accidental happenings.
5	6.*a.*	Without the right breaks one cannot be an effective leader.
	b.	Capable people who fail to become leaders have not taken advantage of their opportunities.
6	7.*a.*	No matter how hard you try some people just don't like you.
	b.	People who can't get others to like them don't understand how to get along with others.
7	9.*a.*	I have often found that what is going to happen will happen.
	b.	Trusting to fate has never turned out as well for me as making a decision to take a definite course of action.
8	10.a.	In the case of the well prepared student there is rarely if ever such a thing as an unfair test.
	b.	Many times exam questions tend to be so unrelated to course work that studying is really useless.
9	11.a.	Becoming a success is a matter of hard work, luck has little or nothing to do with it.
	b.	Getting a good job depends mainly on being in the right place at the right time.
10	12.a.	The average citizen can have an influence in government decisions.
	b.	This work is run by the few people in power, and there is not much the little guy can do about it.
11	13.a.	When I make plans, I am almost certain that I can make them work.
	b.	It is not always wise to plan too far ahead because many things turn out to be a matter of good or bad fortune anyhow.
12	15.a.	In my case getting what I want has little or nothing to do with luck.
	b.	Many times we might just as well decide what to do by flipping a coin.
13	16.*a.*	Who gets to be the boss often depends on who was lucky enough to be in the right place first.
	b.	Getting people to do the right thing depends upon ability, luck has little or nothing to do with it.
14	17.*a.*	As far as world affairs are concerned, most of us are the victims of forces we can neither understand, nor control.
	b.	By taking an active part in political and social affairs the people can control events.
15	18.*a.*	Most people don't realize the extent to which their lives are controlled by accidental happenings.
	b.	There really is no such thing as "luck".
16	20.*a.*	It is hard to know whether or not a person really likes you.
	b.	How many friends you have depends upon how nice a person you are.
17	21.*a.*	In the long run the bad things that happen to us are balanced by the good ones.
	b.	Most misfortunes are the result of lack of ability, ignorance, laziness, or all three.
18	22.a.	With enough effort we can wipe out political corruption.
	b.	It is difficult for people to have much control over the things politicians do in office.
19	23.*a.*	Sometimes I can't understand how teachers arrive at the grades they give.
	b.	There is a direct connection between how hard I study and the grades I get.
20	25.*a.*	Many times I feel that I have little influence over the things that happen to me.
	b.	It is impossible for me to believe that chance or luck plays an important role in my life.

TABLE 2 CONTINUED

21	26.a.	People are lonely because they don't try to be friendly.
	b.	There's not much use in trying too hard to please people, if they like you, they like you.
22	28.a.	What happens to me is my own doing.
	b.	Sometimes I feel that I don't have enough control over the direction my life is taking.
23	29.*a.*	Most of the time I can't understand why politicians behave the way they do.
	b.	In the long run the people are responsible for bad government on a national as well as on a local level.

* The number immediately preceding each item refers to the original ordering. Filler items have been deleted. External choices are italicized.

note that the analysis of Christoffersson [1975] suggests that more than one factor is present for the items of Section 7.

As is seen in Table 1, the estimates obtained by GLS are very similar to those of ML and GLS (old).

The differences between the ML- and the ULS-estimates are also of little significance. The standard errors obtained by GLS are very close to those of GLS (old), reflecting the fact that the estimators are asymptotically as efficient. For both estimators the results are close to those of the ML estimator.

We will now consider a larger data set, which gives us an opportunity to compare the computing times of the two GLS estimators. Thus, the main purpose is not to give a complete analysis of these data. The data set consists of 23 items from a personality inventory described in Rotter [1966]. There are two sub-sets, the first with 201 subjects from the University of Cincinnati, the second with 190 subjects from the 11th and 12th grades of a high school in Baltimore. I am obliged to Mr. David Brandt and Professor R. Darrell Bock at the University of Chicago for making these data available. For the present purposes the two sets of data will be combined, yielding a total of 391 cases. The wording of the 23 items is given in Table 2. The items are concerned with the subject's expectations about how reinforcement, or reward, is controlled. When the subject believes that a reinforcement follows from some action of his own, but is not entirely dependent on this, the subject's belief is labelled *external control.* If the subject believes that the reinforcement is contingent upon his/her own action, the belief is labelled *internal control.* An internal response is coded 1; an external response is coded 0.

For a preliminary analysis the ULS estimator was used. In an exploratory way, the solutions for 1, 2, 3 and 4 factors were obtained. In each case we applied an unrestricted model. From inspection of the loading matrices it was decided to adopt the 3-factor solution. Let $\mathbf{T}(p \times p)$ be the matrix of sample tetrachoric correlations (with unit diagonal elements). A rough way to determine the number of factors is to look for a sharp break between the size of the characteristic roots of $\mathbf{T} - \hat{\mathbf{\Psi}}$, where $\hat{\mathbf{\Psi}}$ contains the estimated specific variances. For the 3-factor solution the ten largest roots of this matrix were: 3.37, 1.47, .96, .79, .55, .42, .31, .27, .21, .17. In our opinion, this suggests that in addition to a few major factors, there are several factors of minor importance involved. Thus, as a next step of analysis we selected items which seemed to represent the 3 factors well. These items can be tested for a 3-factor structure by means of the chi-square measure of fit given by the GLS estimator. The following 15 items were selected: 1, 2, 3, 6, 9, 10, 13, 14, 15, 16, 17, 18,

TABLE 3

Promax-rotated loading matrix and factor correlation matrix using GLS and ULS*, with 15 items selected from Rotter [1966]. †

Item	Loading matrix Factor 1	Factor 2	Factor 3	Specific variance
1	.38 (.41)	-.15 (-.10)	.27 (.16)	.72 (.80)
2	-.11 (-.12)	.46 (.37)	.09 (.17)	.77 (.80)
3	.08 (.03)	.28 (.15)	.23 (.31)	.81 (.85)
6	-.05 (.11)	.00 (-.06)	.70 (.69)	.53 (.51)
9	.66 (.55)	.07 (.12)	-.11 (-.08)	.59 (.68)
10	.00 (-.07)	.71 (.74)	-.16 (-.17)	.52 (.52)
13	.78 (.50)	.19 (.20)	-.22 (-.11)	.42 (.70)
14	.03 (-.05)	.54 (.59)	.06 (.07)	.68 (.63)
15	.63 (.63)	-.13 (-.09)	.04 (-.03)	.59 (.62)
16	-.05 (.01)	-.04 (.04)	.51 (.43)	.76 (.82)
17	.44 (.57)	-.17 (-.19)	.32 (.26)	.61 (.59)
18	.11 (.09)	.66 (.62)	.07 (.06)	.51 (.56)
20	.46 (.43)	.17 (.13)	.02 (.02)	.73 (.78)
21	-.25 (-.15)	.09 (.06)	.52 (.45)	.74 (.78)
23	.09 (.14)	.42 (.35)	.11 (.10)	.76 (.81)

Factor correlation matrix

Factor 1	1.00		
Factor 2	.12 (.17)	1.00	
Factor 3	.37 (.18)	.24 (.33)	1.00

* ULS-estimates are given in parentheses.
† The numbering refers to that of Table 2.

20, 21, 23. Again, an unrestricted model was employed. A chi-square of 74.90 with 63 degrees of freedom (p = .145) was obtained. Since the sample is rather small, we regard this as bordering on significant lack of fit. In addition to the present 3 factors, some of the items probably have some minor factors in common. This test implies that the whole battery of 23 items also has at least 3 factors. In Table 3 the promax-rotated factor loading matrix and the corresponding factor correlation matrix are given. For comparison, the ULS solution is also given. We note that Factors 1 and 2 apparently have to do with perceived control on the individual level and in political matters, respectively. Factor 3 has to do with perceived appreciation. From Table 3 we find that the ULS estimator gives an average communality of about 30%, while GLS yields an additional 5%.

The present GLS estimator and the GLS (old) estimator demand about equal computing time for the initial calculations preceding the minimization (only done once for each data set). In the 15 item case this time was about 85 seconds, using an IBM 370/155 with FORTRAN H compiler. For GLS (old) the minimization took about 190 seconds. With the same starting values and about the same number of iterations, GLS demanded approximately 90 seconds. This is a cut in time with about 53%. The gain would of course be even larger if we had also obtained the solutions for some number of factors other than 3. We may note that the computing time of GLS is rapidly decreasing with decreasing number of items. In the 10 item case the total time is usually less than half a minute and in the 5 item case it is only a few seconds. For ULS, the total computing time was about 10 seconds.

Discussion

An important result in Christoffersson [1975] was the demonstration that very little efficiency in estimation seems to be lost using information from a few marginal proportions relative to using all possible 2^p proportions, as in the maximum likelihood estimation. Even if computable, the extra work of the maximum likelihood method may not be worthwhile. With the proposed GLS estimator we are almost back to the simple traditional method of fitting the factor model. It is however advantageous to use GLS whenever this is computationally feasible. Besides being a more efficient estimator, it readily gives a statistical test of model fit.

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