0.1 Latent variable interactions

Structural equation modeling with latent variable interactions has been discussed with respect to maximum-likelihood estimation in Klein and Moosbrugger (2000). Multivariate normality is assumed for the latent variables. The ML computations are heavier than for models without latent variable interactions because numerical integration is needed. For an overview of the ML approach and various estimators suggested in earlier work, see Marsh et al. (2004), Arminger and Muthén (1998), Klein and Muthén (2007), Cudeck et al. (2009), and Mooijaart and Bentler (2010) discuss alternative estimators and algorithms. This section discusses interpretation, model testing, explained variance, standardization, and plotting of effects for models with latent variable interactions.

0.1.1 Model interpretation

As an example, consider the latent variable interaction model of Figure 1. The figure specifies that the factor $\eta_3$ is regressed on $\eta_1$ and $\eta_2$ as well as the interaction between $\eta_1$ and $\eta_2$, as shown by the structural equation

$$\eta_3 = \beta_1 \eta_1 + \beta_2 \eta_2 + \beta_3 \eta_1 \times \eta_2 + \zeta_3.$$  

(1)

The interaction variable $\eta_1 \times \eta_2$ involves only one parameter, the slope $\beta_3$. The interaction variable does not have a mean or a variance parameter. It does not have parameters for covariances with other variables. It can also not be a dependent variable. As is seen in Figure 1, the model also contains a second structural equation where $\eta_4$ is linearly regressed on $\eta_3$, so that there is no direct effect on $\eta_4$ from $\eta_1$ and $\eta_2$, or their interaction.

For ease of interpretation the (1) regression can be re-written in the equivalent form

$$\eta_3 = (\beta_1 + \beta_3 \eta_2) \eta_1 + \beta_2 \eta_2 + \zeta_3.$$  

(2)

where $(\beta_1 + \beta_3 \eta_2)$ is a moderator function (Klein & Moosbrugger, 2000) so that the $\beta_1$ strength of influence of $\eta_1$ on $\eta_3$ is moderated by $\beta_3 \eta_2$. The choice of moderator when translating (1) to (2) is arbitrary from an algebraic point of view, and is purely a choice based on ease of substantive interpretation. As an example, Cudeck et al. (2009) considers school achievement ($\eta_3$) influenced by general reasoning ($\eta_1$), quantitative ability ($\eta_2$), and their interaction. In line with (2) the interaction is expressed
as quantitative ability moderating the influence of general reasoning on school achievement. Plotting of interactions further aids the interpretation as discussed in Section 0.1.5.

0.1.2 Model testing

As pointed out in Mooijaart and Satorra (2009), the likelihood-ratio $\chi^2$ obtained by ML for models without latent variable interactions is not sensitive to incorrectly leaving out latent variable interactions. For example, the model of Figure 1 without the interaction term $\beta_3 \eta_1 \times \eta_2$ fits data generated as in (1) perfectly. This is due to general maximum-likelihood results on robustness to non-normality (Satorra, 1992, 2002). Misfit can be detected only by considering higher-order moments than the second-order variances and covariates of the outcomes. Without involving higher-order moments, a reasonable modeling strategy is to first fit a model without interactions and obtain a good fit in terms of the ML likelihood-ratio $\chi^2$. An interaction term can then be added and the $\beta_3$ significance of the interaction significance tested by either a $z$-test or a likelihood-ratio $\chi^2$ difference test (Klein & Moosbrugger, 2000). Likelihood-ratio or Wald tests can be used to test the joint significance of several interaction terms.
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0.1.3 Explained variance

The variance and R-square of $\eta_3$ in (1) can be expressed using the following results for moments of normal variables. Assuming multivariate normality for four random variables $x_i, x_j, x_k, x_l$ any third-order moment about the mean ($\mu$) is zero (see, e.g., Anderson, 1984),

$$E(x_i - \mu_i)(x_j - \mu_j)(x_k - \mu_k) = 0,$$

while the fourth-order moment about the mean is a function of covariances,

$$E(x_i - \mu_i)(x_j - \mu_j)(x_k - \mu_k)(x_l - \mu_l) = \sigma_{ij} \sigma_{kl} + \sigma_{ik} \sigma_{jl} + \sigma_{il} \sigma_{jk}. \quad (4)$$

Assuming bivariate normality and zero means for $\eta_1$ and $\eta_2$, all third-order moments $E(\eta_i \eta_j \eta_k)$ are therefore zero so that applying (4) to (1) the variance of $\eta_3$ is

$$V(\eta_3) = \beta_1^2 V(\eta_1) + \beta_2^2 V(\eta_2) + 2 \beta_1 \beta_2 Cov(\eta_1, \eta_2) + \beta_3^2 V(\eta_1 \times \eta_2) + V(\zeta_3), \quad (5)$$

where

$$V(\eta_1 \times \eta_2) = E(\eta_1 \eta_2 \eta_1 \eta_2) - [E(\eta_1 \eta_2)]^2 \quad (6)$$

$$= V(\eta_1) V(\eta_2) + 2 Cov(\eta_1, \eta_2)^2 - [Cov(\eta_1, \eta_2)]^2 \quad (7)$$

$$= V(\eta_1) V(\eta_2) + [Cov(\eta_1, \eta_2)]^2. \quad (8)$$

R-square for $\eta_3$ can be expressed as usual as

$$\frac{[V(\eta_3) - V(\zeta_3)]}{V(\eta_3)}. \quad (9)$$

Using (5) the proportion of $V(\eta_3)$ contributed by the interaction term can be quantified as (cf. Mooijaart & Satorra, 2009; p. 445)

$$\beta_3^2 \frac{[V(\eta_1) V(\eta_2) + [Cov(\eta_1, \eta_2)]^2]}{V(\eta_3)}. \quad (10)$$

Consider as a hypothetical example the latent variable interaction model of Figure 2. Here, the latent variable interaction is between an exogenous and an endogenous latent variable. This example is useful to study the details of how to portray the model. The structural equations are

$$\eta_1 = \beta \eta_2 + \zeta_1, \quad (11)$$

$$\eta_3 = \beta_1 \eta_1 + \beta_2 \eta_2 + \beta_3 \eta_1 \times \eta_2 + \zeta_3. \quad (12)$$

Let $\beta = 1$, $\beta_1 = 0.5$, $\beta_2 = 0.7$, $\beta_3 = 0.4$, $V(\eta_2) = 1$, $V(\zeta_1) = 1$, and $V(\zeta_3) = 1$. This implies that $V(\eta_1) = \beta^2 V(\eta_2) + V(\zeta_1) = 1^2 \times 1 + 1 = 2$ and $Cov(\eta_1, \eta_2) = \beta V(\eta_2) = 1 \times 1 = 1$. Using (5), $V(\eta_3) = 3.17$. The $\eta_3$ R-square is 0.68 and the variance percentage due to the interaction is 15%.
0.1.4 Standardization

Because latent variables have arbitrary metrics, it is useful to also present interaction effects in terms of standardized latent variables. Noting that (12) is identical to (1), the model interpretation is aided by considering the moderator function \((\beta_1 + \beta_3 \eta_2) \eta_1\) of (2) and standardizing with respect to the three latent variables. As usual, standardization is obtained by dividing by the standard deviation of the dependent variable and multiplying by the standard deviation of the independent variable.

The standardized \(\beta_1\) and \(\beta_3\) coefficients in the term \((\beta_1 + \beta_3 \eta_2)\) are obtained by dividing both by \(\sqrt{V(\eta_3)} = \sqrt{3.17}\), multiplying \(\beta_1\) by \(\sqrt{V(\eta_1)} = \sqrt{2}\), and multiplying \(\beta_3\) by \(\sqrt{V(\eta_1)} \sqrt{V(\eta_2)} = \sqrt{2}\). This gives a standardized \(\beta_1 = 0.199\) and a standardized \(\beta_3 = 0.159\). The change in \(\eta_3\) as a function of a change in \(\eta_1\) can now be evaluated at different values of \(\eta_2\) using the moderator function. At the zero mean of \(\eta_2\), a standard deviation increase in \(\eta_1\) leads to a 0.199 standard deviation increase in \(\eta_3\). At one standard deviation above the mean of \(\eta_2\), a standard deviation increase in \(\eta_1\) leads
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to a $0.199 + 0.159 \times 1 = 0.358$ standard deviation increase in $\eta_3$. At one standard deviation below the mean of $\eta_2$, a standard deviation increase in $\eta_1$ leads to a $0.199 - 0.159 \times 1 = 0.04$ standard deviation increase in $\eta_3$. In other words, the biggest effect of $\eta_1$ on $\eta_3$ occurs for subjects with high values on $\eta_2$.

0.1.5 Plotting of interactions

The interaction can be plotted as in Figure 3. Using asterisks to denote standardization, consider the rearranged (12),

$$\eta_3^* = (\beta_1^* + \beta_3^* \eta_2^*) \eta_1^* + \beta_2^* \eta_2^* + \zeta_3^*.$$  \hfill (13)

Using (eq:beta-star moderator fcn), the three lines in the figure are expressed as follows in terms of the conditional expectation function for $\eta_3^*$ at the three levels of $\eta_2^*$,

$$E(\eta_3^*|\eta_1^*, \eta_2^* = 0) = \beta_1^* \eta_1^*,$$  \hfill (14)

$$E(\eta_3^*|\eta_1^*, \eta_2^* = 1) = (\beta_1^* + \beta_3^*) \eta_1^* + \beta_2^*.$$  \hfill (15)

$$E(\eta_3^*|\eta_1^*, \eta_2^* = -1) = (\beta_1^* - \beta_3^*) \eta_1^* - \beta_2^*.$$  \hfill (16)

Here, the standardized value $\beta_2^* = \beta_2 \times \sqrt{V(\eta_2)}/\sqrt{V(\eta_3)} = 0.7 \times 1/\sqrt{3.17} = 0.393$. 

Figure 3: Interaction plot for structural equation model with interaction between an exogenous and an endogenous latent variable.