# Computing the Strictly Positive Satorra-Bentler Chi-Square Test in Mplus

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# 1 Introduction

In this note we illustrate with several examples how to compute the strictly positive robust chi-square test described in Satorra and Bentler (2010). This robust chi-square can be used as an alternative to the robust chi-square proposed in Satorra and Bentler (2001) which can sometimes produce negative test statistics due to its asymptotic nature. We will illustrate this computation using a simple factor analysis model estimated with the MLR estimator in Mplus. Suppose that we have a model $M_1$ and and a more restricted model $M_0$. We denote by $F_i$ the robust test of fit for model $M_i$, by $L_i$ the log-likelihood value for model $M_i$, by $c_i$ the correction factor for $F_i$, by $b_i$ the correction factor for $L_i$, $d_i$ the degrees of freedom for $F_i$, and $p_i$ the number of parameters in model $M_i$. All of these quantities can be easily found in the Mplus output. In Mplus the standard robust chi-square for testing $M_0$ against $M_1$ can be computed in two different ways, using the test of fit and its correction factor, or using the log-likelihood and its correction factor. The two lead to the same result. One of the advantages of the log-likelihood approach is that it does not require the existence of a test of fit and it can be used for any pair of nested models. Using the test of fit the robust chi-square for testing $M_0$ against $M_1$ is computed as follows

$$F = \frac{(F_0c_0 - F_1c_1)(d_0 - d_1)}{c_0d_0 - c_1d_1} \tag{1}$$

Using the log-likelihood values the robust chi-square for testing $M_0$ against $M_1$ is computed as follows

$$F = \frac{2(L_1 - L_0)(p_1 - p_0)}{b_1p_1 - b_0p_0} \tag{2}$$

Both versions produce exactly the same result (up to a round off error). The quantity

$$F_0c_0 - F_1c_1 = 2(L_1 - L_0)$$

is the unadjusted chi-square statistic while

$$\frac{c_0d_0 - c_1d_1}{d_0 - d_1} = \frac{b_1p_1 - b_0p_0}{p_1 - p_0} \tag{3}$$

is the correction factor for testing $M_0$ against $M_1$. In the above formulas the numerators are guaranteed to be positive however the denominators are not, i.e., in all applications we have

$$F_0 c_0 > F_1 c_1$$

$$L_1 > L_0$$

$$d_0 > d_1$$

$$p_1 > p_0$$

but in some applications the asymptotic inequalities

$$c_0 d_0 > c_1 d_1$$

$$b_1 p_1 > b_0 p_0$$

will not hold. To avoid this problem the Satorra and Bentler (2010) use a new test of fit correction factor $c_{10}$ or a likelihood correction factor $b_{10}$ which are obtained from a model $M_{10}$. This model $M_{10}$ can be viewed as unoptimized $M_1$ model having the $M_0$ parameter estimates. More specifically the model $M_{10}$ is the same as the $M_1$ model except that its parameter estimates have not been optimized to convergence but instead its parameter estimates are those of the $M_0$ model. In addition the log-likelihood value of $M_{10}$ is the same as that of $M_0$. With these quantities the new Satorra and Bentler (2010) chi-square is computed as follows

$$F^* = \frac{(F_0 c_0 - F_1 c_1)(d_0 - d_1)}{c_0 d_0 - c_{10} d_1} \tag{4}$$

or equivalently using the log-likelihood values

$$F^* = \frac{2(L_1 - L_0)(p_1 - p_0)}{b_{10} p_1 - b_0 p_0}, \tag{5}$$

where $c_{10}$ and $b_{10}$ are the correction factors obtained from the model $M_{10}$ estimation while all other quantities are as in formulas (1) and (2). In this improved version the denominators are guaranteed to be positive, i.e.,

$$c_0 d_0 > c_{10} d_1$$

$$b_{10} p_1 > b_0 p_0$$

not just asymptotically but also for small sample sizes.

It is important to focus not just on the fact that the new test has a positive value, but what really is at stake is the quality of the results measured by the type I error of the test and the power of the test. These issues are however beyond the scope of this note. We can just mention that alternative tests such as those obtained with the estimators BAYES, MLM, MLMV, WLSM, WLSMV, ULSMV, could actually yield more accurate results than both the new and the regular Satorra-Bentler tests. All of these test statistics are solid asymptotically however there is little guidance for small sample size situations. Usually a simulation study can reveal the quality of the results in small sample size cases.

The new Satorra-Bentler test should be used instead of the traditional version when the test statistic is negative. However, there are probably other situations when the new version should be used. For example when the correction factor (3) is very small (almost negative) we should probably be using the new version as well. This line of arguments leads to the problem of when to actually compute the new SB test statistic. More research is needed to answer this question.

## 2 Example 1

Consider the factor analysis model with 5 indicator variables and 1 latent factor

$$Y_j = \nu_j + \lambda_j \eta + \varepsilon_j$$

where $Y_j$ for $j = 1, ..., 5$ are the observed variables, $\eta$ is an unobserved normally distributed latent variable, $\varepsilon_j$ are zero mean normally distributed residuals with a variance covariance $\Theta$. For identification purposes $\lambda_1 = 1$. In model $M_0$ we estimate the following parameters: 5 parameters $\nu_j$, $j = 1, ..., 5$; 4 loading parameters $\lambda_j$, $j = 2, ..., 5$; the variance of $\eta$ parameter $\psi$, the 5 residual variance parameters $\theta_{jj}$, $j = 1, ..., 5$; and a residual covariance parameter $\theta_{12}$ for a total of 16 parameters. In model $M_1$ in addition to the $M_0$ parameters we estimate the residual covariance $\theta_{23}$. The results of the $M_0$ and $M_1$ estimation using a data set with 50 non-normal observations are presented in Table 1. Now using formula (1) we get that $F = -1.420$. Using formula (2) we get $F = -1.435$. The small difference between the two versions is due to the round off error in the values that Mplus reports (up to the third digit after the decimal point). Thus this example illustrates the problems with the standard robust chi-square which leads to a negative value.

Table 1: Results for models $M_0$ and $M_1$.

| Model | $M_0$ | $M_1$ |
|---|---|---|
| Test of fit $F_i$ | 18.902 | 7.301 |
| Degrees of freedom $d_i$ | 4 | 3 |
| Correction factor for test of fit $c_i$ | 0.515 | 1.095 |
| Log-likelihood $L_i$ | -250.888 | -250.015 |
| Number of estimated parameters $p_i$ | 16 | 17 |
| Correction factor for log-likelihood $b_i$ | 1.843 | 1.663 |

In contrast, when we estimate $M_{10}$ we obtain $c_{10} = 0.034$ and $b_{10} = 1.850$ and when we use the new chi-square $F^*$ using formula (4) we get $F^* = 0.889$ and using formula (5) we get $F^* = 0.890$. Again the small difference is due to the round off error. Our illustration confirms that the test statistic $F^*$ yields positive values. This result is also close to the T-value test statistic based on the robust standard errors for the additional parameter $\psi_{23}$: $(T - value)^2 = (0.823)^2 = 0.677$.

# 3 Practical Aspects of the Estimation

The main issue in the estimation is how to construct an Mplus run that yields the correction factors $c_{10}$ and $b_{10}$. The first step in doing so is actually in the estimation of model $M_0$. Using the SVALUES option of the OUTPUT command we obtain the model $M_{10}$ written in the Mplus language. Copy and paste that model specification into the input file used for the $M_1$ estimation. You will need to add the command for the extra parameters in model $M_1$, in our example that is the command $Y_2\ WITH\ Y_3^*0$; which sets a starting value for the $\theta_{23}$ parameter. Once we have this model setup for $M_{10}$ we need to make Mplus estimate all quantities with 0 iterations, i.e., for this exact same parameters. This is achieved by specifying the option CONVERGENCE=100000000 in the ANALYSIS command[1]. This option specifies the

[1]For some models the command that accomplishes this is MITER=1. These are the models that are estimated with the EM algorithm such as mixture models and models that require numerical integration. Both options can be used if it is not clear which option is appropriate. Alternatively, the commands MCONVERGENCE=100000000; ALGO=ODLL;

convergence criterion in Mplus which is typically 0.00005. The result of this command is that in the very first attempt to optimize the parameters, the log-likelihood derivatives will be computed and will be compared against the convergence value. Because the convergence value is so large Mplus will conclude that convergence has already been achieved and that the parameters are optimized. Then Mplus will provide the needed output which includes the correction factors $c_{10}$ and $b_{10}$. When running the model $M_{10}$ include the TECH5 option of the OUTPUT command. This option will show all iterations that have been taken. We want to make sure that no iterations are taken in that run. Usually there are some iterations in that run with the heading "TECHNICAL OUTPUT FROM EM ALGORITHM ITERATIONS FOR THE H1 MODEL". Those iterations do not count as they are used for computing the sample statistics. You should expect to see no iterations in the sections "ITERATIONS USING GRADIENT" and "ITERATIONS USING QUASI-NEWTON". In addition one should check that the log-likelihood value of the $M_{10}$ model is the same as that for the $M_0$ model.

Finally we want to point out that the example we presented here is a bit simpler than the examples used in practical situations such as tests of invariance and tests of partial invariance. In such more complicated examples additional steps may be necessary to specify the model $M_{10}$. The basic definition of this model is that it has the parameterization of $M_1$, i.e., is parameterized as an $M_1$ model and it has the same number of parameters as the $M_1$ model but it has the same log-likelihood value as the $M_0$ model. The model $M_{10}$ always exists. Its existence is guaranteed by the fact that $M_0$ is nested within $M_1$. To construct the $M_{10}$ model one simply needs to understand how and why exactly $M_0$ is nested within $M_1$. Three further examples of how to construct $M_{10}$ are given in Sections 4 - 6.

The input files for the three models used in this example are presented below in Appendix A.

# 4 Example 2

In certain cases the robust test of fit itself, that is, the test against a completely unrestricted model, will have the problem of negative chi-square val-

can be used for this estimation. The OUTPUT options TECH5 and TECH8 can be used to verify that the parameter estimates are not altered and no maximization iterations occur.

ues. Consider as an example the same model as in Example 1 with added residual covariance parameters $\theta_{12}$, $\theta_{23}$, $\theta_{45}$, $\theta_{15}$. Here the model $M_1$ is the unrestricted mean and variance model. The test of fit for this model yields a negative value. To construct the $M_{10}$ model in this case we can use the RESIDUAL option of the OUTPUT command to obtain the estimated mean and variance covariance for the observed variables $Y_j$ for the $M_0$ model. To setup $M_{10}$ we setup an unrestricted model for $Y_j$ where all means, variance and covariance starting values are set at the estimated quantities from the $M_0$ model. In this situation only formula (5) can be used. From the 3 runs we get $L_0 = -249.800$, $L_1 = -246.017$, $b_0 = 1.721$, $b_1 = 1.644$, $p_0 = 19$ and $p_1 = 20$. Using (5) we get the test of fit value $F^* = 41.801$.

# 5 Example 3

In some cases the construction of $M_{10}$ involves non-standard approaches because the difference between $M_0$ and $M_1$ is not as simple as having an additional parameters in $M_1$. Consider as an example a factor analysis model as in Example 1, without any residual correlations, but conducted as a two group model. The standard setup for this $M_0$ model is that the the residual variances for the observed variables and the factor variance are group specific, the observed variable intercepts are group invariant as well as the loading parameters, and the factor mean is estimated as a free parameter in the second group. Suppose that we need to test the loading invariance across the two groups, i.e., we consider the more flexible model where the loadings are not held equal across the two groups. To construct the $M_{10}$ model in this case we simply need to use the SVALUES option of the OUTPUT command in the $M_0$ estimation, copy and paste this model in the $M_1$ input file and remove the parameter labels that hold the loadings equal across the two groups.

# 6 Example 4

In some cases the construction of $M_{10}$ involves actual computations that reveal how exactly the $M_0$ model is nested within the $M_1$ model. Consider for example the $M_0$ model we used in the previous section and the $M_1$ model which has varying across group observed variables intercepts, which necessarily eliminates also the factor means from the second group for identification purposes. To construct the $M_{10}$ model in this case we use both the SVALUES and RESIDUAL options of the OUTPUT command in the $M_0$ model estimation. The $M_{10}$ construction is started with the model produced by SVALUES but we modify it in three steps. Step 1 would be to fix the factor mean in the second group to 0. Step 2 is to remove the equality labels in the two groups for the intercept parameters. Step 3 is to replace the starting values for the intercept parameters in the second group with the estimated values found in the RESIDUAL output in the $M_0$ model estimation. This example illustrates how quickly the complexity of the construction of the $M_{10}$ model can escalate. In many cases the values that are needed will not be available in the RESIDUAL output, for example if there is a covariate in the above model.

# 7 Appendix A: Mplus input for example 1

**Inputs for the other examples are available at**
**http:\\www.statmodel.com\examples\webnote.shtml # web12**

```
title: M0 model
variable: names are y1-y5;
data: file=1.dat;
analysis: estimator=mlr;
model: f1 by y1-y5; y1 with y2;
output: svalues;
```

```
title: M1 model
variable: names are y1-y5;
data: file=1.dat;
analysis: estimator=mlr;
model: f1 by y1-y5; y1 with y2; y3 with y2;
```

```
title: M10 model
variable: names are y1-y5;
data: file=1.dat;
analysis: estimator=mlr; convergence=100000000;
model: y3 WITH y2*0;
f1 BY y1@1;
f1 BY y2*0.599;
f1 BY y3*0.814;
f1 BY y4*0.893;
f1 BY y5*0.818;
y1 WITH y2*-0.031;
[ y1*0.280 ];
[ y2*0.380 ];
[ y3*0.400 ];
[ y4*0.420 ];
[ y5*0.420 ];
y1*0.167;
y2*0.337;
y3*0.378;
y4*0.385;
y5*0.420;
f1*0.274;
output: tech5;
```

# References

[1] **Satorra, A., & Bentler, P.M. (2010) Ensuring Positiveness of the Scaled Difference Chi-square Test Statistic. Psychometrika, 75, 243-248. http:\\www.springerlink.com\content\k716217434q71737\fulltext.pdf**

[2] **Satorra, A., & Bentler, P.M. (2001). A scaled difference chi-square test statistic for moment structure analysis. Psychometrika, 66, 507-514.**